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A STUDY OF COMPRESSIBLE PERFECT FLUID MOTION IN TURBOMACHINES WITH INFINITELY MANY BLADES

BY

GERALD MORGAN MONROE

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M681

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A STUDY OF COMPRESSIBLE PERFECT FLUID MOTION IN TURBOMACHINES WITH INFINITELY MANY BLADES

Thesis by

Gerald Morgan Monroe

Lieutenant, U. S. Navy

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1951

Thesis
M681

Dedicated to my wife

JOYCE

For her patience and understanding

throughout the years of my graduate studies.

ACKNOWLEDGMENT

It is a pleasure to acknowledge the interest and assistance offered by members of the Aeronautics staff of the California Institute of Technology during the entire course of my research. I am particularly indebted to Dr. Frank E. Marble, my research advisor, whose friendship and personal encouragement have strongly influenced the course of all my work, and Dr. H. S. Tsien, whose clear insight animated several fruitful discussions. I should like to express my deep appreciation to Dr. W. M. Coates for his inspiring confidence, to Professor M. H. Vavra for his early direction to the salient features of my research problem, and to all other members of the U. S. Naval Postgraduate School for their guidance during the first phases of my graduate work.

I am most grateful for the policies of the Bureau of Aeronautics of the Navy Department which made it possible for me to pursue graduate studies while on active duty in the U. S. Navy.

I was ably aided in the rather extensive calculations by Mrs. Eloise Skavdahl and the computing staff of the GALCIT Wind Tunnel. The numerous figures and charts were prepared in final form by Mrs. Betty Wood. The tedious task of typing the thesis was most excellently performed by Mrs. Elizabeth Fox.

ABSTRACT

A study is made of compressible perfect fluid motion in turbomachines having infinitely many blades and a general theory is developed. An underlying concept of the theory is that force fields which represent the action of infinitely many blades belong to a special class described as pseudo-conservative and can be expressed as the product of a scalar function and the gradient of a potential. The scalar function is simply the rate at which energy is imparted to the fluid by the blades, and the potential is simply the family of the equations for the blade surfaces. The introduction of these two functions to express the force field casts an entirely new light on problems of mixed-flow turbomachines having infinitely many blades of arbitrary shape.

In the formulation of the problem the non-linear action of rotationality and compressibility is regarded as a force tending to displace the streamsurfaces from their irrotational, incompressible position. It is shown that the character of the problem is determined by a governing velocity: the velocity relative to the blades where blades are present, or the meridional velocity, where blades are not present. Where the governing velocity is subsonic the problem is essentially elliptic, where supersonic, hyperbolic.

The theory and the examples lead to conclusions which are believed to explain in part the unexpected efficiencies observed for compressors having transonic governing velocities. These conclusions, which indicate that transonic compressors could perhaps be profitably developed, are as follows: The deflection of the streamsurfaces induced by a given strength of vorticity at a certain point in the flow has one sense when the governing velocity at the point is subsonic, the opposite sense when it is supersonic, and becomes zero as it becomes sonic. The deflection of the streamsurfaces brought about by a given distribution of vorticity in a region is less when the governing velocity in the region is transonic than when it is entirely subsonic or entirely supersonic.

Examples of incompressible flow through a mixed flow compressor with prescribed blades, and subsonic and transonic flow through actuator disks, were solved by the method of finite differences, applying simultaneously the relaxation technique and an iteration process.

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I. INTRODUCTION

In a real turbomachine a viscous, compressible fluid flows through an axially symmetrical channel. In a region of this channel a system of blades, either stationary or rotating about the axis of symmetry, acts on the fluid. The field of the flow, being bounded by the surfaces of the blades as well as by the channel boundaries, is not circumferentially uniform. The total energy of the fluid may vary from point to point in the field, the flow is generally rotational, and if the blade system rotates, the flow is unsteady. The real problem is thus a very formidable one, and cannot be approached exactly by methods known today. If viscous effects are neglected the problem is greatly simplified. Because of this, and because viscous effects actually are very slight except near the boundaries, the assumption of zero viscosity has always been made.

A brief but excellent review of the earlier important investigations of the flow in turbomachines is given by Marble (Reference 1). Earlier investigations were concerned primarily with the flow through a typical annulus of small radial extent and hence treated the flow as essentially two-dimensional, e. g. Traupel (Reference 2). Interference between the flow in neighboring annular regions was assumed negligible, a condition which is fulfilled only if the centrifugal forces and the radial pressure gradient forces are in balance, that is, if

the flow is that of a vortex centered on the axis of rotation. This is a rather severe restriction, too strong to be imposed in most applications.

The first detailed analysis of the three-dimensional incompressible flow in turbomachines was given by Meyer (Reference 3). An exact solution was obtained for machines in which the flow is irrotational upstream and downstream of the rotor, although the flow within the rotor may be rotational. Meyer achieved circumferential uniformity by assuming an infinite number of very thin blades, and then modified this result by a Fourier analysis to obtain the solution for a finite number of blades. This was an important contribution but the extension to arbitrary blade shape, with rotational flow downstream of the rotor, has not been made.

Marble (References 1 and 4) linearized the rotational incompressible problem with an infinite number of blades by the assumption that the vorticity is transported along the streamlines of an irrotational flow within the same boundaries. He provided examples of axial flow and conical flow, and stated that the simple linearized solution was sufficiently accurate if the vorticity effects were not large. A second order linearization was given to handle problems in which the vorticity effects are large. Using the simple linearizations Marble considered three interesting problems: the mutual interference of neighboring blade rows in a multistage axial turbo-

rotors, the solution for a straight [illegible] of given shape, and [illegible] for this blade row operating at a condition different from the design point.

The "two-dimensional" compressible flow in a centrifugal compressor with a finite number of straight blades was investigated by Stanitz and Ellis (reference 5). Flow was considered in a narrow passage the center-line of which generated a right circular cone when rotated about the axis of the compressor. The two-dimensional flow pattern lay on this cone, and the flow conditions were assumed to be uniform across the passage normal to the conical surface. The analysis yielded the first detailed information about compressible flow between blades with finite spacing, but three-dimensional effects were not included, and only the flow between straight blades was examined.

The present analysis is concerned with an idealized problem that differs from the real problem only in that the fluid is inviscid and the number of blades is infinite. The fluid may be compressible but the occurrence of shock waves is excluded. The more general mixed-flow, where both the axial and radial velocity components may be large, is considered. Included are the special cases of axial-flow, where the axial velocity component is much larger than the radial component, and radial-flow, where the radial velocity component is much larger than the axial component. The shape of the

blades is completely arbitrary. The two assumptions which make the analysis possible are: the fluid considered is non-viscous, and the number of blades is infinite. The viscosity of the fluid is unimportant except within the boundary layers along all the bounding surfaces, and these boundary layers are thin if pressure gradients are favorable. Furthermore, an elementary consideration of boundary layers often presupposes a knowledge of the velocity distributions such as obtained by this analysis. The assumption that the number of blades is infinite makes the flow field circumferentially uniform. This uniformity makes the solution easier in that the flow depends only on two coordinates, but at the same time it introduces artificial complications. Generally there will be a discontinuity in the flow as it enters this blade region if the number of blades is infinite. Furthermore the force of the blades on the fluid is not applied on individual blade surfaces, but is distributed throughout the blade region and actually acts as a body force field distributed throughout the region. The circumferentially uniform flow field may be considered as the limiting case of a flow acted upon by a very large number of closely spaced blades of negligible thickness.

Two distinct problems occur in practise. The design problem arises when it is desired to design a machine for a particular purpose. The analysis problem arises when it is desired to

investigate a given machine under given operating conditions. These problems were classified by Marble (Reference 4) as the indirect and the direct problem, respectively, in analogy with the three-dimensional wing theory. This classification is convenient and very significant. Here any problem in which the blade shape is prescribed will be classified as a direct problem, any other problem as an indirect problem. The full significance of this distinction becomes clear when the circumstances under which the idealized problem is actually comparable to a real problem are understood.

In the real machine, which has a finite number of blades, each blade transmits a force to the fluid by maintaining a discontinuity in pressure across its two surfaces. If no viscous force is present this blade force must be normal to the blade surface. In idealizing the problem these concentrated blade forces are essentially replaced by a body force field. Clearly this body force field must be normal to the family of blade surfaces throughout the region of the blades if the two problems are to be comparable. It is important to note that this imposes a purely geometrical restriction on the body force field. It will be shown that the necessary and sufficient condition that it be possible to construct a family of surfaces which are everywhere perpendicular to a given force field is that the force field is everywhere perpendicular to its own curl. If the blade shape is prescribed, as in the direct problem, this offers no difficulty.

However in the indirect problem where the blade shape is one of the things to be determined, the prescribed quantities, which for example, may be components of the force field or the energy distribution, must be prescribed in such a way that the existence of the family of blade surfaces is assured beforehand. This necessity* was noted by Meyer (Reference 3) in connection with the prescription of vorticity in the indirect problem, but was overlooked by Reissner (Reference 6), Gravalos (Reference 7), and several others. When the streamwise extent of the working region is much less than its transverse extent, that is, if the blade aspect ratio is large, as in most axial flow problems, the idealized flow will be comparable to the real flow even though the blade existence has not been assured. Thus Marble (Reference 4) was able to obtain good approximate solutions of the indirect axial flow problem by prescribing blade force components. However in mixed-flow problems the aspect ratio is not large, and in the indirect mixed-flow problem the existence of the blades must be assured beforehand.

In the real problem one boundary condition is that the fluid does not flow through the blade surfaces. This cannot be applied as a boundary condition in the idealized problem since the blade surfaces, being infinite in number, are not boundaries of the flow.

---

*Information lately received reveals that this condition for integrability of the blade surfaces was stated by Bauersfeld as early as 1905 in a paper (Reference 10) not available at the time of writing.

instead the fluid in the *region* of the blades is constrained to movement on the blade surfaces. This constraint, quite different from the boundary condition of the real problem, plays an important part in the mathematical formulation of the idealized problem. The geometrical relation which expresses the constraint is equal in significance to the other equations and must be considered simultaneously with them. In addition, with uniform inlet conditions and an incompressible fluid, the differential equation for the flow is linear in the region of the blades, although it is generally non-linear downstream of this region. This fact enabled Meyer (Reference 3) to find exact solutions for flows in which the vorticity downstream of the blades is zero, although the flow may be rotational and very complicated in the region of the blades.

To summarize, the idealized problem must satisfy the equations of motion--with the body force field included, the continuity equation, the isentropic pressure-density relation, and those boundary conditions of the real problem not applied on the blade surfaces. The two additional requirements which must be fulfilled if the solution of the idealized problem is to be comparable to the real problem are: the body force field must be normal to the family of blade surfaces; and the velocity vector field must be parallel to the family of blade surfaces.

In the direct problem the blade shape is known and these

supplementary conditions can easily be imposed. For the indirect problem in which force components are prescribed these requirements should be expressed in terms of the force components as follows: the body force field must be normal to the velocity vector field; and the force vector must be normal to its own curl. These assure the existence of a family of surfaces which can be chosen to fulfill the two requirements stated above for the direct problem. If in the indirect problem other quantities such as energy distribution are prescribed, equivalent requirements must be satisfied.

The idealized problem is formulated in terms of a stream function for the velocities in the meridional plane, and the resulting differential equation for the stream function is written as a second order non-homogeneous partial differential equation by regarding the non-linear terms, representing the effects of rotationality and compressibility, as forcing functions tending to displace the streamsurfaces. The differential equation is then replaced by a finite difference equation which is solved by a simultaneous application of the relaxation technique of Southwell (Reference 8) and an iteration process.

The general theory is developed in Parts II through VI, and the difference formulation and examples are presented in Parts VII, VIII, IX, and X. The examples were conceived in order to demonstrate separately the different phases of the problem. Part

VIII is concerned with incompressible flow through a region of prescribed blades and is primarily an example of rotationality in a mixed-flow compressor. Part IX is offered as an example of the effects of compressibility when the vorticity distribution is approximately constant. Part X is an example of the interaction of rotationality and compressibility effects when the total velocity is transonic but the governing velocity is subsonic.

II. COORDINATES, NOTATION, SYMBOLS

The flow is described (Figure 1) for the most part in an absolute cylindrical coordinate system r, θ, z by the velocity components u, v, w respectively. The absolute velocity vector is:

$$\bar{V} = \bar{i}_r u + \bar{i}_\theta v + \bar{i}_z w ,$$

where $\bar{i}_r$, $\bar{i}_\theta$, and $\bar{i}_z$ are unit vectors in the r, θ, and z directions respectively. The vorticity components for circumferentially uniform flow are respectively:

$$\xi = -\frac{\partial v}{\partial z} \; ; \; \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \; ; \; \zeta = \frac{1}{r}\frac{\partial rv}{\partial r} \qquad (1)$$

and the vorticity vector, $\bar{\Omega} = \nabla \times \bar{V}$, is:

$$\bar{\Omega} = \bar{i}_r \xi + \bar{i}_\theta \eta + \bar{i}_z \zeta$$

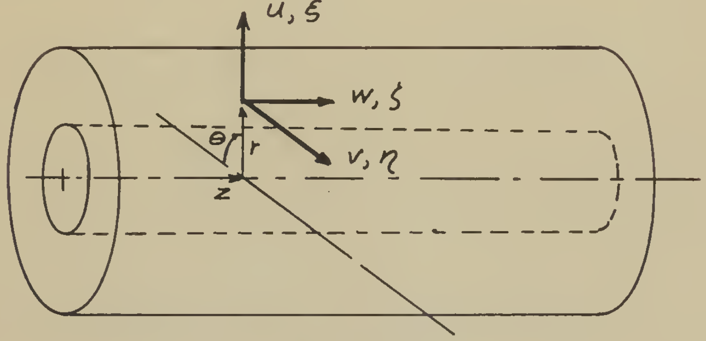


Figure 1
The Velocity and the Vorticity Components in the Absolute Coordinate System

The scalar functions u, v, w, ξ, η, ζ are independent of the θ coordinate in accordance with the requirement of circumferential uniformity.

The force of the blades on the fluid is given by

$$\bar{F} = \bar{i}_r F_r + \bar{i}_\theta F_\theta + \bar{i}_z F_z$$

and has the dimensions of force/unit mass, or $\frac{\text{length}}{\text{time}^2}$.

At times the relative cylindrical coordinates r, ϕ, z, where ϕ is measured with respect to the rotor, will be used. If the angular velocity of the rotor is ω, the absolute and relative velocities are related by:

$$d\theta - d\phi = \omega\, dt \qquad (2)$$

The velocity and vorticity vectors in the relative system are respectively:

$$\bar{W} = \bar{V} - \bar{i}_\phi \omega r = \bar{i}_r u + \bar{i}_\phi (v - \omega r) + \bar{i}_z w \;, \qquad \text{and}$$

$$\bar{\Omega} = \bar{\Omega} - \bar{i}_z 2\omega = \bar{i}_r \xi + \bar{i}_\phi \eta + \bar{i}_z (\zeta - 2\omega)$$

The intrinsic coordinates n, θ, s will often permit a more concise presentation (Figure 2). n is distance normal to the meridional streamline and s is distance along the meridional streamline. The velocity and vorticity vectors are respectively:

$$\bar{V} = \bar{i}_\theta v + \bar{i}_s q_s \;, \qquad \text{and}$$

$$\bar{\Omega} = \bar{i}_n \Omega_n + \bar{i}_\theta \eta + \bar{i}_s \Omega_s \;,$$

where

$$\Omega_n = -\frac{1}{r}\frac{\partial rv}{\partial s}\;, \quad \Omega_s = \frac{1}{r}\frac{\partial rv}{\partial n}\;,$$

$$\eta = -K_s q_s - \frac{\partial q_s}{\partial n} \qquad (3)$$

and where K_s is the curvature of the meridional streamline.

The family of the blade surfaces is given by a scalar function

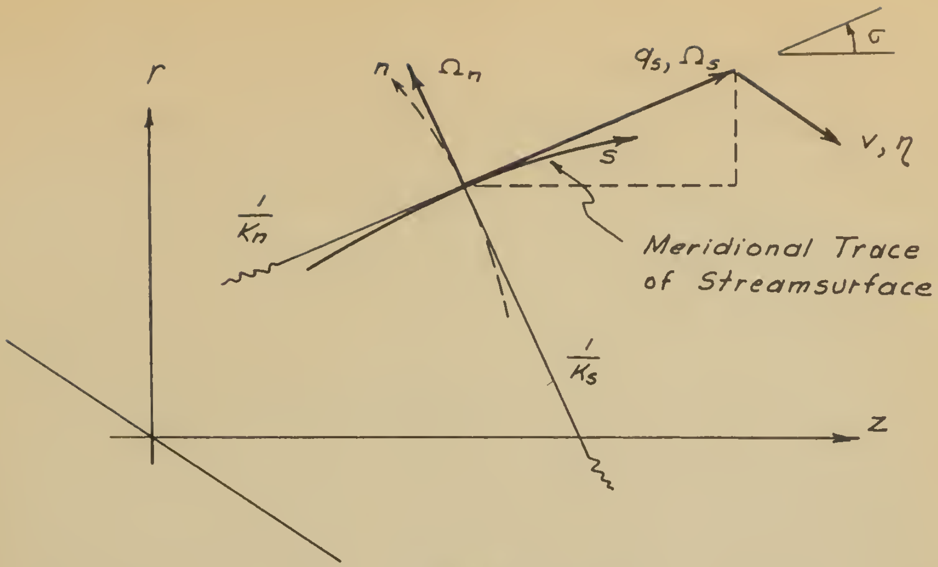


Figure 2

The Velocity and the Vorticity Components in the Intrinsic Coordinate System

of position, called the blade surface function:

$$\beta(r,\theta,z) = \theta - f(r,z) \tag{4}$$

A blade surface is defined by β = constant. The function $f(r,z) = (\theta - \beta)$ defines the family of the traces of the blades in a meridional plane. The normal to the blade surface is

$$\nabla\beta = -\bar{i}_r f_r + \bar{i}_\theta \frac{1}{r} - \bar{i}_z f_z \tag{5}$$

Since the blade force $\bar{F}$ is always normal to the instantaneous surface of the blades it can be expressed as

$$\bar{F} = \lambda(r,z)\, \nabla\beta(r,\theta,z) \tag{6}$$

This defines the scalar function $\lambda(r,z)$. From the second equation of motion it will be shown later that $\omega\lambda$ is the local rate at which energy is added to the fluid by the action of the blade forces. β and f are non-dimensional, $\nabla\beta$ has dimensions of $(\text{length})^{-1}$, and

has dimension of $\frac{\text{length}^2}{\text{time}^2}$.

Note that $\nabla\beta$, the normal to the blade surface, is not a unit vector. Instead it is chosen so that its tangential component is unity. Thus the magnitude of the tangential component of the force vector is simply $F_\theta = \frac{1}{r}\lambda$.

Miscellaneous symbols:

R = $r\frac{\partial f}{\partial r}$

Z = $r\frac{\partial f}{\partial z}$

p = static pressure

ρ = density

a = speed of sound

γ = ratio of specific heats

t = time

E = $\frac{1}{2}V^2 + \int\frac{dp}{\rho}$ = total energy of the fluid

$(\ )_1$ subscript denotes condition far upstream of the region of the blades, or at the upstream boundary

$(\ )_t$ subscript denotes condition at the trailing edge of the blades

$(\ )_r$ $(\ )_z$, partial differentiation with respect to r and z respectively, unless otherwise defined

a_r = angle between the blade surface and the meridional plane measured in a plane where z is constant

a_z = angle between the blade surface and the meridional plane measured in a plane tangent to the surface where r is constant

a_n = angle between the blade surface and the meridional plane measured in a plane normal to their line of intersection

K_s = curvature of the meridional stream line

K_n = curvature of the normal to the meridional streamline

q_s = meridional velocity component

$\mathcal{R}$ = "residual" of the difference operator

$D(\ )$ = difference operator

Γ = circulation about an annular vortex ring

σ = angle between meridional velocity and axis

a_{ij} = influence coefficients

M = Mach number

r_o = outer radius

r_i = inner radius

$\tilde{r} = \dfrac{r - r_i}{r_o - r_i}$ = non-dimensional radius

III. THE PSEUDO-CONSERVATIVE FORCE FIELD AND THE BLADE SURFACE FUNCTION

A) The Necessary and Sufficient Condition for the Existence of the Family of Blade Surfaces

The body force field $\bar{F}$ is normal to the family of blade surfaces β, hence $\bar{F}$ and $\nabla\beta$ are parallel and can be related* by

$$\bar{F}(r,z) = \lambda(r,z)\ \nabla\beta(r,\theta,z) \tag{6}$$

where $\lambda(r,z)$ is a scalar function, and $\nabla\beta$ is a function of only r and z because of circumferential uniformity. Writing

$$\left(\frac{1}{\lambda}\right)\bar{F} = \nabla\beta$$

and taking the curl of both sides there results:

$$\left(\frac{1}{\lambda}\right)\nabla\times\bar{F} + \nabla\left(\frac{1}{\lambda}\right)\times\bar{F} = 0$$

Then forming the scalar products with $\bar{F}$,

$$\left(\frac{1}{\lambda}\right)\ \bar{F}\cdot\nabla\times\bar{F} = 0$$

and since λ is not zero unless $|\bar{F}|$ is zero.

$$\bar{F}\cdot\nabla\times\bar{F} = 0 \tag{7}$$

This is the necessary condition for the existence of the family of blade surfaces and is the additional relation which must be fulfilled in the indirect problem if the idealized problem is to be comparable to a real problem.

In order to discuss the question of blade existence suppose that in the indirect problem force components are to be prescribed. First consider flow through the channel with no force present. The flow is completely determined by the equations of motion, the con-

---

* The applicability of Equation (6) and the subsequent derivation of Equation (7) were pointed out by Dr. Arthur Erdélyi in personal conversation.

tinuity equation, a pressure-density relation, and the necessary boundary conditions. Now consider flow through the same channel with forces present. Three unknowns, the three force components, have been added to the problem. One equation, the requirement that the relative streamline be normal to the force field, has been added:

$$\overline{W} \cdot \overline{F} = 0 \tag{8}$$

At this point it appears that two components of the force can be prescribed arbitrarily. The third component would then be expressed by means of Equation (8), and the problem would be completely determined as in the channel with no force acting. The solution would describe a flow under the action of the prescribed forces, but would not necessarily be comparable to a flow acted upon by blades because the existence of the blade surfaces has not been assured. If Equation (7) is imposed the problem is redundant, for with two force components prescribed there are more equations than unknowns. An example will give physical meaning to this difficulty. Suppose the two components F_r and F_θ are prescribed as:

$$F_r = 0 \; ; \quad F_\theta = F_\theta(r,z)$$

F_z can be expressed in terms of these and the velocity by Equation (8) as:

$$\frac{F_z}{F_\theta} = -\frac{v-\omega r}{w} \tag{9}$$

The three force components are then known and the problem is completely prescribed as in the channel with no force acting. The solution will give u, v, w as functions of r and z. Now the equation as to the existence of blade surfaces is raised. Equation (7) for this case is

$$-F_\theta \frac{\partial F_z}{\partial r} + F_z \frac{1}{r} \frac{\partial r F_\theta}{\partial r} = 0,$$

from which

$$\frac{F_z}{r F_\theta} = \text{function of z only} \tag{10}$$

The problem is clearly over prescribed, for F_z is restricted when Equation (10) is imposed. Furthermore, it is clear what should be prescribed. If F_r is zero, then $\frac{F_z}{rF_\theta}$ can be a prescribed function of z only. From Equations (9) and (10),

$$\frac{v-\omega r}{w} = -\frac{F_z}{F_\theta} = -r \text{ x function of z}$$

This function of z that may be prescribed represents an angle between the relative velocity and the meridional plane and actually is the blade shape parameter $f_z(z)$ defined below. Thus when the existence of blades is assured the indirect problem with force components properly prescribed is equivalent to the direct problem with the blade shape prescribed.

B) The Most General Form of the Body Force Field

The reason that the two force components cannot be independently prescribed is that the force field, although non-conservative, is restricted in form. The most general force field that will satisfy Equation (7) is given by Equation (6). This is the most general force

field for which a family of blade surfaces actually exists and is therefore the most general force field possible in turbomachine problems where the number of blades is infinite. For this reason it is proposed for all such problems that the body force field be expressed as the product of a scalar function and the gradient of a potential:

$$\bar{F} = \lambda \nabla \beta = \lambda \nabla [\theta - f(r,z)] \tag{11}$$

This force field has a special place between conservative fields and non-conservative fields. A conservative force field can be expressed in terms of one scalar function, its potential. A non-conservative force field requires three scalar functions for expression, one for each component. The force field of the turbomachine, being intermediate between these in that it requires two scalar functions for expression, might be called a pseudo-conservative force field. Actually its vectors have the same direction as a conservative field, but not the same magnitude.

This fact that the force field depends on only two scalar functions indicates why, in theindirect problem with force components prescribed, the two components cannot be prescribed independently. It also indicates that considerable simplification might result from considering the functions λ and f instead of the three force components. This is actually the case. The introduction of these two functions for the force vector field casts an entirely new light on turbomachine problems in which the number of blades is

infinite. For example, the heretofore more difficult direct problem becomes relatively simple. Furthermore the functions λ and f have special physical significance: $\omega\lambda(r,z)$ represents the local time rate at which energy is added to the fluid by the action of rotating blades, and f(r,z) describes the blade shape completely.

From Equation (11) the force field is finally expressed as

$$\bar{F} = \lambda\left(-\bar{i}_r f_r + \bar{i}_\theta \frac{1}{r} - \bar{i}_z f_z\right) \qquad (12)$$

When the force field is expressed in terms of the functions λ and f the question of blade existence does not arise and the direct and indirect problems become quite clear. A direct problem is one in which the blade shape f(r,z) is prescribed and $\lambda(r,z)$ is an unknown determined by the second equation of motion after the solution is complete. The function λ does not appear in the equation for the meridional flow. An indirect problem may be one in which the local rate of energy input, $\lambda(r,z)$ is prescribed, and the blade surface function, f(r,z), is an unknown dependent variable.

C) The Physical Meaning of the Blade Surface Function β :

It is of interest to note the geometrical significance of β and f and their derivatives. The absolute value of $\nabla\beta$ is

$$|\nabla\beta| = \frac{1}{r}\left[1 + R^2 + Z^2\right]^{\frac{1}{2}} \qquad (13)$$

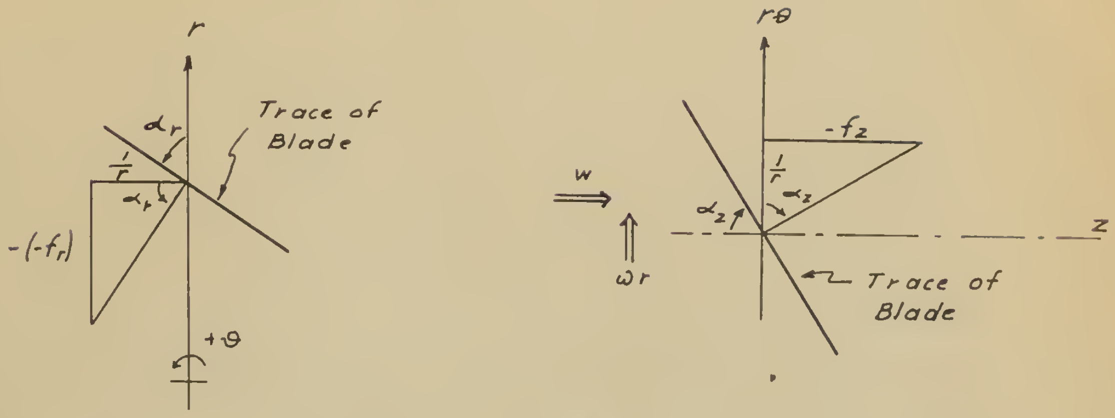
where $R = rf_r$ and $Z = rf_z$

The relation between the vector $\nabla\beta$ and the blade surface is shown in Figure 3. Using the definitions of α_r, α_z, α_n as given in Part

11 it easily follows that

$$R = \tan \alpha_r \qquad (14)$$
$$Z = -\tan \alpha_z$$

with the sign convention shown in Fig. 3.



Looking Downstream onto a Plane z = constant

Looking Radially Inward onto a Plane Tangent to a Surface r = constant

Figure 3
The Blade Surface Function f(r,z)

In the same way

$$R^2 + Z^2 = \tan^2 \alpha_n \qquad (15)$$

and from Equation (13)

$$r/\nabla\beta/ = \sec.\alpha_n \qquad (16)$$

Thus R and Z are merely the tangents of certain angles between the blade surface and the meridional plane.

The functions β and f were defined in such a way that β = constant is the equation of a blade surface and f = constant is the equation of the trace of a blade surface in the meridional plane.

If the blades are so-called radial blades, that is, if the blade surfaces are generated by lines that are normal to the axis, the function f is independent of r. In this case $f_r = 0$ and f_z is a function of z only.

IV. THE HYDRODYNAMICAL EQUATIONS

A) The Equations in the Various Coordinate Systems

The hydrodynamical equations for the steady, adiabatic, axially symmetrical, and circumferentially uniform flow of a non-viscous fluid acted upon by a pseudo-conservative* body force field are given in the various coordinate systems previously described.

Vector Forms:

The equation of motion:

$$\bar{V} \cdot \nabla \bar{V} = -\frac{1}{\rho} \nabla p + \lambda \nabla \beta \qquad (17a)$$

$$\bar{V} \times \bar{\Omega} = \nabla \frac{1}{2} \bar{V} \cdot \bar{V} + \frac{1}{\rho} \nabla p - \lambda \nabla \beta \qquad (17b)$$

The continuity equation:

$$\nabla \cdot \rho \bar{V} = 0 \qquad (18)$$

The isentropy condition:

$$\nabla \left(\frac{p}{\rho^{\gamma}} \right) = 0 : \quad a^2 = \frac{dp}{d\rho} = (\gamma - 1) \int \frac{dp}{\rho} \sim \rho^{\gamma - 1} \qquad (19)$$

Absolute Cylindrical Coordinates:

The equations of motion:

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} - \lambda f_r \qquad (20a)$$

$$u \frac{\partial rv}{\partial r} + w \frac{\partial rv}{\partial z} = \lambda \qquad (20b)$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \lambda f_z \qquad (20c)$$

The continuity equation:

$$\frac{1}{r} \frac{\partial}{\partial r} (\rho r u) + \frac{\partial}{\partial z} (\rho w) = 0 \qquad (21)$$

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* A pseudo-conservative ~~forces~~ force field is defined as one which can be expressed as the product of a scalar function and the gradient of a potential. See Part III.

Absolute Intrinsic (Flow) Coordinates:

The equations of motion:

$$q_s \frac{\partial q_s}{\partial s} - \frac{v^2}{r} \sin \sigma = -\frac{1}{\rho} \frac{\partial p}{\partial s} - \lambda f_s \tag{22a}$$

$$q_s \frac{\partial rv}{\partial s} = \lambda \tag{22b}$$

$$K_s q_s^2 - \frac{v^2}{r} \cos \sigma = -\frac{1}{\rho} \frac{\partial p}{\partial n} - \lambda f_n \tag{22c}$$

The continuity equation:

$$\frac{\partial}{\partial s} \rho q_s + \rho q_s K_n + \rho q_s \frac{1}{r} \sin \sigma = 0 \tag{23}$$

B) An Integral of the Equation of Motion

Since the flow is steady the equation of motion can be integrated along each streamline. Forming the scalar product of $\overline{V}$ and Equation (17b), a scalar equation is obtained:

$$\overline{V} \cdot \nabla \frac{1}{2} \overline{V} \cdot \overline{V} + \frac{1}{\rho} \overline{V} \cdot \nabla p - \lambda \overline{V} \cdot \nabla \beta = 0 \tag{24}$$

From Equations (6) and (8) the condition that the relative velocity vector and the force vector are perpendicular is

$$\overline{W} \cdot \nabla \beta = \left(\overline{V} - \overline{i}_\theta \omega r \right) \cdot \nabla \beta = 0 \tag{25}$$

Consequently

$$\overline{V} \cdot \nabla \beta = \omega , \qquad \text{and} \quad v - \omega r = r f_r u + r f_z w \tag{26}$$

From the second equation of motion (Equation (20b)):

$$\lambda = u \frac{\partial rv}{\partial r} + w \frac{\partial rv}{\partial z} = \overline{V} \cdot \nabla (rv) \tag{27}$$

Since the flow is isentropic the density is a function of pressure only, $\rho = \rho(p)$, and the pressure term of Equation (24) is:

$$\frac{1}{\rho} \nabla p = \nabla \int \frac{dp}{\rho} \tag{28}$$

Using Equations (26), (27), and (28), Equation (24) becomes

$$\overline{V} \cdot \nabla \left[\frac{1}{2} V^2 + \int \frac{dp}{\rho} - \omega r v \right] = 0 \qquad (29)$$

Integrating along the meridional streamlines an energy equation analogous to the Bernoulli equation is obtained.

$$\frac{1}{2} \overline{V}^2 + \int \frac{dp}{\rho} - \omega r v \quad = \text{constant on each streamsurface} \qquad (30)$$

Upstream and downstream of the blades rv is constant on each streamsurface. The total energy is $E = \frac{1}{2} V^2 + \int \frac{dp}{\rho}$. The term $\frac{1}{2} V^2$ is the kinetic energy per unit mass and $\int \frac{dp}{\rho}$ is the enthalpy. The change of ωrv which occurs along a streamsurface in the region where the blades act is the change in the total energy of the fluid occurring on the same streamsurface.

The constant in Equation (30) will have the same value on all streamsurfaces if the total energy is uniform and the meridional component of vorticity is zero upstream of the blades. Under these conditions Equation (30) becomes

$$\frac{1}{2} V^2 + \int \frac{dp}{\rho} - \omega r v \quad = \text{constant} \qquad (31)$$

In a more general case the flow may be rotational and may have a non-uniform energy distribution at the upstream boundary, the inlet. Using the subscript $(\ )_1$ to denote these inlet conditions the "constants" in Equation (30) can be evaluated.

$$\frac{1}{2} V^2 + \int \frac{dp}{\rho} - \omega r v = E_1 - \omega (r v)_1 \qquad (32)$$

The functions E_1 and $\omega(rv)_1$ are constant on each streamsurface and generally each has a different value on each streamsurface. They

cannot be evaluated as functions of position unless the streamsurfaces are known.

The gradient of Equation (32) is taken in order to eliminate the total energy gradient, ∇E, from the equation of motion:

$$\nabla E = \nabla \omega r v + \nabla E_1 - \nabla \omega (rv)_1 \qquad (33)$$

Since the functions E_1 and $(rv)_1$ are constants on each streamsurface their gradients are normal to the streamsurface and Equation (33) can be written as

$$\nabla E = \nabla \omega r v + \bar{i}_n \frac{\partial E_1}{\partial n} - \bar{i}_n \, \omega \frac{\partial (rv)_1}{\partial n} \qquad (34)$$

where n is distance normal to the streamsurface.

C) Two Identities

Two identities which can be proved for axially symmetrical flow using only the definition of vorticity are:

$$\bar{\Omega} \cdot \nabla \omega r v = 0 \qquad (35)$$

$$\nabla \omega r v = \bar{i}_\theta \omega r \times \bar{\Omega} \qquad (36)$$

The vector $\bar{i}_\theta \omega r$ has the dimension of velocity and acts tangentially.

D) The Concept of Free and Bound Vorticity

The concept of free and bound vorticity provides a useful means of discussing the properties of rotational flow. Although rotational flow problems are generally non-linear, they may be linear if the vorticity is bound. The non-linearity arises physically from the dependence of the solution upon the distribution of vorticity

which is determined, in turn, by the manner in which the vorticity is transported by the velocities of the solution. Some knowledge of the vorticity field can be obtained directly from the equations of motion.

Consider first rotational flow in a region where no forces act on the fluid. The equation of motion is simply (from Equation (17b)):

$$\overline{V} \times \overline{\Omega} = \nabla \frac{1}{2} V^2 + \frac{1}{\rho} \nabla p = \nabla E \tag{37}$$

If the total energy is uniform on the upstream boundary it is uniform everywhere in the field and $\nabla E \equiv 0$. This is the simplest example of free vorticity, one in which the vorticity vector and the velocity vectors are parallel*. If the total energy of the fluid is not uniform, the velocity and vorticity vectors are not parallel. They are, however, perpendicular to the gradient of the total energy.

Consider next rotational flow in a region where a pseudo-conservative force field acts on the fluid. The equation of motion is (from Equation (17b)):

$$\overline{V} \times \overline{\Omega} = \nabla E - \lambda \nabla \beta \tag{38}$$

With Equation (33) this becomes

$$\overline{V} \times \overline{\Omega} = \nabla \omega r v - \nabla \omega (rv)_1 + \nabla E_1 - \lambda \nabla \beta \tag{39}$$

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*Two parallel vectors do not necessarily have the same sense.

where E_1 and $(rv)_1$ describe conditions prescribed at the inlet and are constant on each streamsurface. Using Equation (36) this becomes at once:

$$(\bar{V} - \bar{i}_\theta \omega r) \times \bar{\Omega} = \bar{W} \times \bar{\Omega} = \nabla E_1 - \nabla \omega (rv)_1 - \lambda \nabla \beta \tag{40}$$

If inlet conditions are uniform ∇E_1 and $\nabla (rv)_1$ are zero and all of the vorticity is generated by the blades. In this case Equation (40) states that the vorticity vector, as well as the relative velocity vector, lies in the blade surface β = constant; hence the term bound vorticity. This is the simplest example of bound vorticity.

Forming the scalar product of the vorticity $\bar{\Omega}$ and Equation (40), the left side is zero:

$$\bar{\Omega} \cdot \nabla \beta = \bar{\Omega} \cdot \frac{1}{\lambda} \nabla [E_1 - \omega (rv)_1] \tag{41}$$

Again this shows that the vorticity vector lies in the blade surface if the upstream conditions are uniform.

Consider last the flow downstream of a system of blades. The equation of motion is simply Equation (40) with $\lambda \equiv 0$:

$$\bar{W} \times \bar{\Omega} = \nabla E_1 - \nabla \omega (rv)_1 \tag{42}$$

If the conditions far upstream are uniform, ∇E_1 and $\nabla (rv)_1$ are zero, and the vorticity vector is parallel to the relative velocity vector. Thus vorticity generated by a moving system of blades will be parallel to the velocity measured relative to the moving system, regardless of the shape of the blades, and regardless of the change in energy effected by the blades. This is merely the "upstream" case

again if relative coordinates are used.

The vorticity generated by a system of blades has certain geometrical properties. In the region of the blades in which the vorticity is generated the vorticity is 'bound' and lies in the blade surfaces. Downstream of the blades the vorticity generated by the blades is 'free' and is parallel to the relative velocity.

E) The Transport of Vorticity

If a fluid particle has once acquired a rotation it tends to maintain this rotation as it moves along the streamline. In other words, the vorticity is transported along the streamline with the velocity of the fluid. The magnitude and direction of the vorticity will vary as the fluid expands or contracts, as the velocity vector changes direction, and of course under the action of non-conservative forces. With uniform inlet conditions the equation of motion is given by Equation (39) with ∇E_1 and $\nabla (rv)_1 = 0$. Taking the curl of both sides,

$$\frac{d\bar{\Omega}}{dt} = \bar{V} \cdot \nabla \bar{\Omega} = \bar{\Omega} \cdot \nabla \bar{V} + \nabla \lambda \times \nabla \beta - \bar{\Omega}\, \nabla \cdot \bar{V} \quad . \tag{43}$$

This shows how the time derivative of the vorticity vector depends on the vorticity itself as well as on the velocity and the force field. If the fluid is incompressible the last term of Equation (43) is zero.

Because of axial symmetry, only the tangential vorticity is associated with the radial and axial velocities, while the radial and axial vorticity components are associated with only the tangential

velocity. The tangential vorticity may be regarded as an infinite number of vortex rings centered on the axis. The circulation Γ around each ring is given by η multiplied by the cross-sectional area of the ring. As the radius of the ring increases the cross-sectional area decreases in such a way that the mass of the fluid in the ring remains constant. It follows that for a given ring, Γ is proportional to $\frac{\eta}{r\rho}$ as the ring deforms. The tangential component of Equation (43) expresses the law governing the time rate change of circulation connected with a certain mass of fluid enclosed in an annular vortex ring:

$$\frac{d}{dt}\frac{\eta}{r} = \frac{1}{r}\frac{\partial}{\partial z}\frac{v^2}{r} + (\lambda_r f_z - \lambda_z f_r) - \frac{\eta}{r}\left(\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{\partial w}{\partial z}\right) \qquad (44)$$

From the continuity equation, Equation (21):

$$\frac{1}{r}\frac{\partial}{\partial r} ru + \frac{\partial}{\partial z} w = -\frac{1}{\rho}\frac{d\rho}{dt} = \rho\frac{d}{dt}\frac{1}{\rho} \qquad (45)$$

With this, Equation (44) can be written:

$$r\rho\frac{d}{dt}\frac{\eta}{r\rho} = \frac{\partial}{\partial z}\frac{v^2}{r} + \frac{\partial}{\partial r}\lambda f_z - \frac{\partial}{\partial z}\lambda f_r \qquad (46)$$

The first term on the right is the rate of change in the axial direction of the centrifugal force, which acts radially; the second term is the radial rate of change of the axial blade force; and the third term is the rate of change of the radial blade force in the axial direction. A little thought will show that these terms, in each case, represent moments tending to cause rotation of a fluid particle about a tangential axis. It is in this way that non-conservative forces

tend to effect a change in the circulation around a vortex ring. If the centrifugal force is conservative the first term is zero, and if the blade force is conservative λ is constant, and the last two terms cancel. Then the circulation about the vortex ring is constant.

Vorticity may therefore be considered as a property of the fluid which continually changes character as the fluid moves along the streamsurfaces. This concept of the transport of vorticity by the fluid particles gives physical significance to the non-linear interaction between the vorticity and the velocity and will be useful in understanding the various steps of the iteration process to be developed later.

V. THE MATHEMATICAL FORMULATION OF THE IDEALIZED PROBLEM

A) The Equations, the Variables, and the Boundary Condition

Available for the idealized problem are six equations: the three equations of motion, Equations (20abc); the continuity equation, Equation (21); the isentropy condition, Equation (18); and the geometrical relation that the relative velocity is tangent to the blade surface, Equation (25). These six equations relate seven dependent variables, u, v, w, p, ρ, λ, f,--each of which is a function of the independent variables r and z. Clearly, since there are only six equations, one of these seven must be prescribed. The angular velocity of the rotor, ω, is a constant parameter.

In the direct problem the blade surface function $f(r,z)$ is prescribed and there are six equations for the remaining six dependent variables. The necessary and sufficient boundary conditions are the same as for flow through the same channel with no blades acting except that a sort of Kutta condition must be applied at the trailing surface of the blade region. The boundary conditions will be discussed in more detail after the number of dependent variables has been reduced.

In an indirect problem where the local rate of energy input, $\lambda(r,z)$, is prescribed there are again six equations and six dependent variables. The question of blade existence, so important in the

indirect problem with force components prescribed, does not arise here, for with the force field expressed in terms of the two functions, λ and f, blade existence is assured. The boundary conditions, however, are less understood than in the direct problem, particularly the conditions for f(r,z). It is to be noted also that while the direct problem, with f prescribed, can be formulated as a problem in the meridional plane independent of λ, the indirect problem, with λ prescribed, is not independent of f. The two problems are therefore quite distinct.

The direct problem is formulated below in terms of a streamfunction for the meridional velocity. The discussion of characteristics and general properties in Part VI also refers to the direct problem.

B) The Differential Equations for the Streamfunction

The continuity equation is identically satisfied by the streamfunction ψ, defined[#] by:

$$\frac{u}{a_*} = \frac{\rho_*}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial z} \tag{47}$$

$$\frac{w}{a_*} = -\frac{\rho_*}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial r} \tag{48}$$

For cases in which the fluid is incompressible take $a_* = 1, \frac{\rho_*}{\rho} = 1$.

The equation of motion is

$$\bar{V} \times \bar{\Omega} = \nabla E - \lambda \nabla \beta \tag{38}$$

[#] In this definition a* and ρ* are considered as constants which describe the initial total energy. If the initial total energy distribution is non-uniform, a* and ρ* are constants which describe the total energy at some representative point on the upstream boundary.

Upstream or downstream of the blades the force magnitude λ is zero and since the total energy remains constant on each streamsurface

$$\nabla E = \bar{i}_n \frac{\partial E}{\partial n} \tag{49}$$

Written in intrinsic coordinates Equation (38) becomes

$$\begin{vmatrix} \bar{i}_n & \bar{i}_\theta & \bar{i}_s \\ 0 & v & q_s \\ \Omega_n & \eta & \Omega_s \end{vmatrix} = \bar{i}_n \frac{\partial E}{\partial n} \tag{50}$$

Using the definition of Ω_n and Ω_s , Equation (3), the normal component is

$$\eta q_s = \frac{v}{r} \frac{\partial rv}{\partial n} - \frac{\partial}{\partial n} E \tag{51}$$

The other components show that $\frac{\partial rv}{\partial s}$ is zero.

Since the total energy is constant on each streamsurface, Equation (34) can be replaced by

$$\nabla E = \bar{i}_n \frac{\partial}{\partial n} \omega[(rv)_t - (rv)_1] + \bar{i}_n \frac{\partial E_1}{\partial n} \tag{52}$$

and Equation (51) can be written for the upstream and downstream regions respectively as:

$$\eta q_s = \frac{1}{r^2} (rv)_1 \frac{\partial (rv)_1}{\partial n} - \frac{\partial E_1}{\partial n} \tag{53}$$

$$\eta q_s = \frac{1}{r^2} [(rv)_t - \omega r^2] \frac{\partial (rv)_t}{\partial n} + \omega \frac{\partial (rv)_1}{\partial n} - \frac{\partial E_1}{\partial n} \tag{54}$$

In the region of the blades λ is not zero and the total energy is not constant on each streamsurface. An expression for the tangential vorticity can be obtained from Equation (41):

$$\bar{\Omega} \cdot \nabla\beta = \bar{\Omega} \cdot \frac{1}{\lambda} \bar{i}_n \frac{\partial}{\partial n}[E_1 - \omega (rv)_1] \tag{41}$$

Expressing the left side in cylindrical coordinates and the right side

in intrinsic coordinates there results:

$$-\xi f_r + \eta \frac{1}{r} - \zeta f_z = \Omega_n \frac{1}{\lambda} \frac{\partial}{\partial n}[E_1 - \omega(rv)_1] \tag{55}$$

But $\Omega_n = -\frac{1}{r}\frac{\partial rv}{\partial s}$, Equation (3); and $\lambda = q_s \frac{\partial rv}{\partial s}$, Eq. (22b)

so that

$$\frac{\Omega_n}{\lambda} = \frac{\frac{1}{r}\frac{\partial rv}{\partial s}}{q_s \frac{\partial rv}{\partial s}} = -\frac{1}{r q_s} \tag{56}$$

and Equation (55) becomes

$$\eta = \left(f_z \frac{\partial rv}{\partial r} - f_r \frac{\partial rv}{\partial z}\right) - \frac{1}{q_s}\frac{\partial}{\partial n}[E_1 - \omega(rv)_1] \tag{57}$$

Equation (57) expresses the tangential vorticity η in the region of the blades. It is significant that this relation is independent of the blade force magnitude λ .

All of the three basic equations, Equations (53), (54), and (57), are non-linear if the upstream conditions, E_1 and $\omega(rv)_1$, are non-uniform, or if the fluid is compressible. If the fluid is incompressible and E_1 and $(rv)_1$ are constant, the equation upstream is linear. It is simply $\eta = 0$, but the downstream equation, Equation (54), is still non-linear. However, the equation in the region of the blade, Equation (57), actually is linear if the fluid is incompressible. For, from Equation (25)

$$rv = \omega r^2 + r^2(f_r u + f_z w) \tag{58}$$

so that in the region of the blades

$$\eta = \left(f_z \frac{\partial}{\partial r} - f_r \frac{\partial}{\partial z}\right)\left\{\omega r^2 + r^2(f_r u + f_z w)\right\} - \frac{1}{q_s}\frac{\partial}{\partial n}[E_1 - \omega(rv)_1] \tag{59}$$

Note that the operator $\left(f_z \frac{\partial}{\partial r} - f_r \frac{\partial}{\partial z}\right)$ represents differentiation along the trace of a blade in the meridional plane. Equation (59) is clearly a linear equation for u and w when the last term is zero and f(r, z) is prescribed.

If η is written in terms of the streamfunction in Equations (53), (54), and (59), the three differential equations for the streamfunction are obtained. By direct substitution, using Equations (47) and (48),

$$\begin{aligned}\eta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ &= \frac{\rho_* a_*}{\rho r}\left[\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial \psi}{\partial r}\frac{1}{\rho}\frac{\partial \rho}{\partial r} - \frac{\partial \psi}{\partial z}\frac{1}{\rho}\frac{\partial \rho}{\partial z}\right]\end{aligned} \tag{60}$$

The last two terms can be written more simply after a change of independent variable as

$$-\frac{\partial \psi}{\partial r}\frac{1}{\rho}\frac{\partial \rho}{\partial r} - \frac{\partial \psi}{\partial z}\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{\partial \psi}{\partial n}\frac{1}{\rho}\frac{\partial \rho}{\partial n} \tag{61}$$

The derivatives taken normal to the streamsurfaces of functions which are constant on the streamsurfaces can be written in a very useful form by considering the streamfunction ψ as an independent variable. For instance, since rv is constant on each streamsurface downstream of the blades, $(rv)_t$ depends only on the streamfunction ψ. Therefore

$$\frac{\partial (rv)_t}{\partial n} = \frac{d(rv)_t}{d\psi}\frac{\partial \psi}{\partial n} = \frac{d(rv)_t}{d\psi}\left(-\frac{q_s}{a_*}\frac{\rho}{\rho_*}r\right) \tag{62}$$

and similarly for any function which is constant on the streamsurfaces.

Using the relations developed in Equations (61) and (62) and

the definition of ψ the differential equations for the streamfunction can now be written for the three regions of the flow. Equations (53), (54), and (59) then become:

Upstream of the blades:

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = \psi_n (\ln \rho)_n + \left(\frac{\rho}{\rho_*}\right)^2 \left[r^2 \frac{d}{d\psi} \frac{E_1}{a_*^2} - \frac{1}{2} \frac{d}{d\psi} \left(\frac{rv}{a_*}\right)_1 \right] \tag{63}$$

In the region of the blades:

$$\begin{aligned} &(1+Z^2)\psi_{rr} - 2RZ\psi_{rz} + (1+R^2)\psi_{zz} + (ZZ_r - RZ_z - \tfrac{1}{r})\psi_r - (ZR_r - RR_z)\psi_z \\ &= 2\frac{\omega r}{a_*}\frac{\rho}{\rho_*}Z + [(1+Z^2)\psi_r - RZ\psi_z](\ln\rho)_r + [(1+R^2)\psi_z - RZ\psi_r](\ln\rho)_z \\ &\qquad + \left(\frac{\rho}{\rho_*}\right)^2 r^2 \frac{d}{d\psi}\left[\frac{E_1}{a_*^2} - \frac{\omega (rv)_1}{a_*^2}\right] \end{aligned} \tag{64}$$

where $R = rf_r$, $Z = rf_z$, and the subscripts r and z denote partial differentiation.

Downstream of the blades:

$$\begin{aligned} \psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = {} & \psi_n (\ln\rho)_n - \left(\frac{\rho}{\rho_*}\right)^2 \left[\left(\frac{rv}{a_*}\right)_t - \frac{\omega r^2}{a_*}\right] \frac{d}{d\psi}\left(\frac{rv}{a_*}\right)_t \\ & + \left(\frac{\rho}{\rho_*}\right)^2 \left[r^2 \frac{d}{d\psi}\left(\frac{E_1}{a_*^2}\right) - \frac{\omega r^2}{a_*^2} \frac{d}{d\psi}\left(\frac{rv}{a_*}\right)_1 \right] \end{aligned} \tag{65}$$

The equations are rewritten below for the flow of an incompressible fluid with uniform inlet conditions in order to discuss the rotationality effects separately from the compressibility effects. For the incompressible fluid, a* and $\frac{\rho_*}{\rho}$ are taken as unity.

Upstream of the blades:

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = 0 \tag{66}$$

In the region of the blades:

$$\begin{aligned} &(1+Z^2)\psi_{rr} - 2RZ\psi_{rz} + (1+R^2)\psi_{zz} \\ &\qquad + (ZZ_r - RZ_z - \tfrac{1}{r})\psi_r - (ZR_r - RR_z)\psi_z = 2\omega r Z \end{aligned} \tag{67}$$

Downstream of the blades:

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = -\left[(rv)_t - \omega r^2\right]\frac{d}{d\psi}(rv)_t \qquad (68)$$

With uniform inlet conditions the flow upstream of the blades is irrotational. Equation (66), which is clearly a linear partial differential equation, states simply that the tangential vorticity η is zero. In the region of the blades and downstream of the blades the flow is rotational. However, in the region of the blades the fluid is constrained to movement on the blade surfaces, and the vorticity is 'bound' and lies in the blade surfaces. Because of this constraint the equation for the streamfunction in the region of the blades, Equation (67), is linear when the blade shape functions, R and Z, are prescribed. The equation for the downstream region, Equation (68), is a non-linear equation for ψ because ψ appears as an independent variable on the right side. However if the right side is a known function of position the equation is linear, although non-homogeneous, and can be handled in the same way as Equations (66) and (67). This suggests an iteration process in which the right side is estimated and the equations are solved. A new right side is calculated from this solution and the process repeated until the desired accuracy is obtained. Actually $(rv)_t$ is a function of ψ only and the non-homogeneous part, the right side of Equation (68), can be evaluated as a function of ψ and r for all the downstream region once the flow conditions at the blade trailing edge are determined. There are several reasons why

this iteration process is a practical and relatively simple method of solving this problem. The non-homogeneous part of Equation (68) has a simple physical meaning. The right side is simply $r\eta$, the moment of the tangential vorticity about the axis, and is proportional to $r^2\Gamma$. It was shown that under the action of a conservative force field the circulation Γ remains constant along the streamsurfaces (Equation (46)). This suggests that a good first approximation might be to assume that the circulation is constant on the streamsurfaces of an irrotational flow through the same channel, thus neglecting the interaction of η and the other vorticity components. Actually a much better approximation can be made by evaluating the function $(rv)_t$ as a function of ψ at the blade trailing edge using the blade geometry and the meridional velocity of an irrotational flow through the same channel. Once $(rv)_t$ is a known function of ψ the right side is a known function of ψ and r. Here the interaction of the vorticity is accounted for, but the blade trailing edge conditions, which define $rv(\psi)$, are of course approximate. In the numerical method of finite differences used in the following examples it is only necessary to solve completely the final step of the iteration process. Each step only need be carried far enough to assure an improvement of the estimate of the right side. The application is relatively easy when the relaxation technique is used since the exact status of the solution is clear at all times.

When the fluid is compressible all of the equations are non-linear. The term on the right side, $\psi_n\,(\ln\rho)_n$, represents the entire effect of compressibility if the flow is irrotational. For rotational flow another effect of compressibility is to modify the effects of the rotation. Thus the density ratio $\left(\frac{\rho}{\rho_*}\right)^2$ is a multiplier of the rotationality terms in Equations (63), (64), and (65). Again an iteration process is used, in which the term $\psi_n(\ln\rho)_n$ is estimated and a solution is obtained. This solution is then the basis for a new estimate of the density term, and the process is repeated until the desired accuracy is obtained.

The conditions under which the above iteration processes are convergent have not been rigorously established. For flow in which a certain governing velocity is supersonic convergence is questionable. If stagnation points occur in the fluid the question of convergence is connected with the question of proper boundary conditions and the fact that the streamsurfaces are characteristic surfaces of the flow. Convergence will be discussed separately for each example solution.

The equation for the flow in the region of the blades is somewhat simpler than shown in Equation (64) if radial blades are prescribed. Radial blades, blades which are generated by radial lines passing through the axis of rotation, are necessary in very

high speed machines because of the high centrifugal forces. The trace in the meridional plane of a radial blade is of course a radial line, hence f(r,z) is actually f(z) only. Then for radial blades:

$$R = 0,\ Z = r f_z(z),\ Z_r = f_z(z),\ etc., \tag{69}$$

and Equation (64) becomes:

$$(1+Z^2)\psi_{rr} - (1-Z^2)\frac{1}{r}\psi_r + \psi_{zz} = 2\frac{\omega r}{a_*}\frac{\rho}{\rho_*}Z + (1+Z^2)\psi_r(\ln\rho)_r + \psi_z(\ln\rho)_z + \left(\frac{\rho}{\rho_*}\right)^2 r^2 \frac{d}{d\psi}\left[\frac{E_1}{a_*^2} - \frac{\omega(rv)_1}{a_*^2}\right] \tag{70}$$

The significance of the simplification is that the cross derivative, ψ_{rz}, does not appear when the blades are radial.

The most general form of the differential equations for the streamfunction is that shown in Equations (63), (64), and (65). These are the equations for the isentropic, rotational flow of a compressible fluid acted upon by an infinite number of arbitrary blades. The flow may be rotational and the energy distribution may be non-uniform at the upstream boundary. An energy relation which expresses density in terms of the gradient of the streamfunction and the total energy of the fluid must be considered with these if the fluid is compressible.

C) The Isentropic Energy Equation--Density as a Function of Mass Flow

The energy equation, Equation (32), can be written as

$$\frac{1}{2}V^2 + \frac{a^2}{\gamma-1} = \omega r v + E_1 - \omega(rv)_1 \tag{71}$$

Let a_o be the velocity of sound when the velocity V is zero. The constant a_o will have a different value on each streamsurface if the

energy on the upstream boundary is non-uniform. A non-uniform a_o will be denoted by a prime, e.g. a_o'. Then Equation (71) becomes

$$\frac{a^2}{a_o'^2} = 1 - \frac{\gamma-1}{2}\left\{\frac{V^2 - 2\omega(rv - rv_1)}{a_o'^2}\right\} \tag{72}$$

or in terms of density

$$\frac{\rho}{\rho_o'} = \left[1 - \frac{\gamma-1}{2}\left(\frac{u^2}{a_o'^2} + \frac{w^2}{a_o'^2}\right) - \frac{\gamma-1}{2}\left(\frac{v^2}{a_o'^2} - 2\,\frac{\omega rv - \omega rv_1}{a_o'^2}\right)\right]^{\frac{1}{\gamma-1}} \tag{73}$$

Introducing the streamfunction defined by Equations (47) and (48),

$$\frac{\rho}{\rho_o'} = \left[1 - \frac{\gamma-1}{2}\left(\frac{\rho_o'}{\rho}\right)^2\left(\frac{a_*^2}{a_o'^2}\,\frac{\rho_*^2}{\rho_o'^2}\,\frac{\psi_r^2+\psi_z^2}{r^2}\right) + \left(\frac{v^2 - 2\omega(rv - rv_1)}{a_o'^2}\right)\right]^{\frac{1}{\gamma-1}} \tag{74}$$

This equation gives the density in terms of the local mass flow,

$$\frac{\rho^2 q_s^2}{\rho_o'^2 a_o'^2} = \left(\frac{a_*^2}{a_o'^2}\,\frac{\rho_*^2}{\rho_o'^2}\,\frac{\psi_r^2+\psi_z^2}{r^2}\right) \tag{75}$$

and the effective tangential velocity

$$\frac{V_{eff.}^2}{a_o'^2} = \left(\frac{v^2 - 2\omega(rv - rv_1)}{a_o'^2}\right) \tag{76}$$

This last term includes the effect of changes of total energy as a result of blade action as well as the actual effect of the tangential velocity.

The solution of Equation (74) is presented graphically as computational curves, Figure 42.

Two other useful forms of the energy equation can be derived from Equation (72):

$$\frac{\rho^2}{\rho_o'^2} = \frac{\left(\frac{\rho_*}{\rho_o'}\right)^2 \frac{\psi_r^2+\psi_z^2}{r^2}}{\frac{2}{\gamma-1}\left(\frac{a_o'^2}{a_*^2} - \frac{a^2}{a_*^2}\right) + \frac{2\omega(rv - rv_1) - v^2}{a_*^2}} \tag{77}$$

$$\frac{d\rho}{\rho} = \frac{\frac{1}{2}d\left(\frac{\psi_r^2+\psi_z^2}{r^2}\right)}{\frac{\psi_r^2+\psi_z^2}{r^2}} + \frac{\frac{1}{2}d\left(\frac{2}{\gamma-1}\,\frac{a^2}{a_*^2} - \frac{2\omega(rv - rv_1) - v^2}{a_*^2}\right)}{\frac{2}{\gamma-1}\left(\frac{a_o'^2}{a_*^2} - \frac{a^2}{a_*^2}\right) + \frac{2\omega(rv - rv_1) - v^2}{a_*^2}} \tag{78}$$

D) The Matching of Solutions in Adjoining Regions

There are three distinct regions of the flow corresponding to the three differential equations, Equations (63), (64), and (65), and solutions in adjoining regions must be matched on the connecting boundaries. The streamfunction itself is always continuous across both the leading edge and the trailing edge boundaries. The Kutta condition imposed at the blade trailing edges to make the solution unique requires that the velocity and pressure be continuous there. Consequently, the matching conditions at the trailing edges are that the streamfunction and its first derivatives, as well as the tangential velocity, are continuous across the trailing edges. This provides the means of evaluating the function $(rv)_t$ for the downstream region. Since rv is continuous it may be evaluated just upstream of the trailing edge by Equation (25), which can be written as

$$rv = \omega r^2 + r^2(f_r u + f_z w) \tag{79}$$

The matching conditions at the leading edges are generally much more complicated. In the real problem with a finite number of blades the fluid flows smoothly through the blade system. A "stagnation point" occurs at some point near the leading edge and the fluid flows smoothly off the trailing edge. The flow is similar to the flow about an airfoil. However, in the idealized problem the number of blades is infinite, and the velocity at the leading edge

will generally be discontinuous. The flow is very similar to the flow through an infinitely closely spaced lattice of airfoils, and the leading edge discontinuity is the same as that occurring at the leading edge of the lattice. In problems of practical interest the pressure and tangential velocity will generally be discontinuous at the leading edge. Continuity of the first and higher derivatives of the streamfunction depends on the shape of the blade near the leading edge. It can be shown that the first derivatives of the streamfunction are continuous across the leading edges provided only that the leading edges lie in meridional planes (Figure 4).

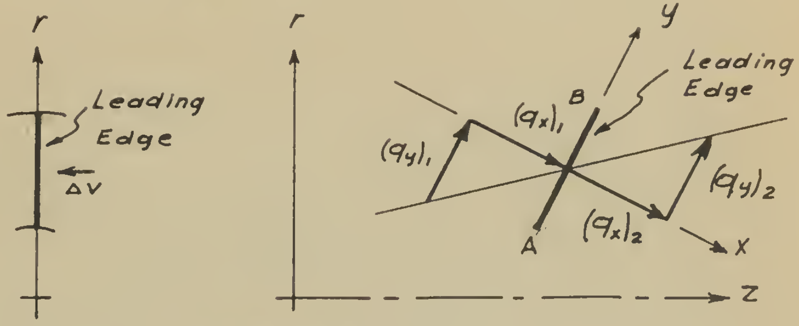

Figure 4
The Flow Across the Blade Leading Edge

It is supposed that the tangential velocity v jumps discontinuously across the leading edge. This implies that a discontinuity in density, pressure, and velocity may also occur there. The first derivatives of the streamfunction are proportional to ρq_s, the local mass-velocity in the directions of the derivatives. ρq_s is of course continuous on both sides of the leading edge, and continuity of mass require that

$(\rho_1)(q_x)_1 = (\rho_2)(q_x)_2$ in Figure 4. Therefore ρq_s is continuous everywhere, and the first derivatives of the streamfunction are continuous everywhere. For blade surfaces which satisfy this condition, the matching conditions at the leading edge are that the streamfunction and its first derivatives are continuous there. The tangential, radial, and axial velocity components and the pressure and density may be discontinuous. If the blades do not satisfy this condition the streamfunction derivatives will generally be discontinuous.

E) The Boundary Conditions

A typical region of flow is shown in Figure 5. The axially

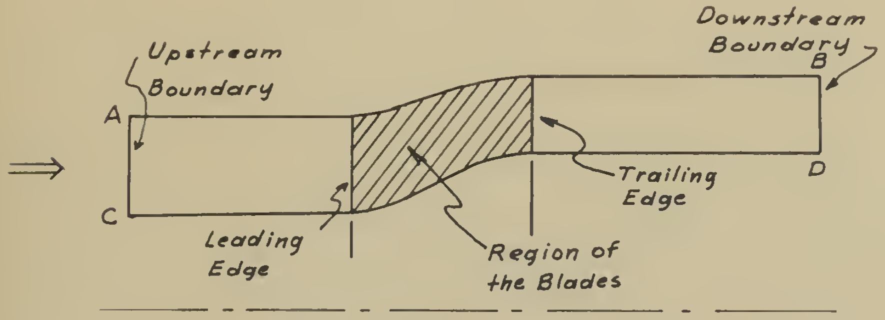

Figure 5
A Typical Region of Flow

symmetrical surfaces AB and CD are boundaries of the channel and are streamlines of the flow. The upstream and downstream boundaries are the surfaces AC and BD respectively. The region of the blades or the body force field, over which the blade surface function f(r,z) is prescribed, is shaded.

For each step of the iteration the equations for the streamfunction, Equations (63), (64), (65), are non-homogeneous linear second order partial differential equations. The necessary and sufficient boundary conditions for this problem are well known. The streamfunction ψ, or a linear combination of its derivatives, ψ_r and ψ_z, must be prescribed at every point on the boundary. The boundary conditions on the streamfunction for a typical region of flow (Figure 5) are:

ψ = constant on surface AB

ψ = constant on surface CD

$\psi = \psi(r)$ on surface AC

$\frac{\partial \psi}{\partial z} = 0$ on surface BD

The functions v, p, and ρ, which can be prescribed only once on each streamsurface, are prescribed at the inlet, on surface AC.

At the leading edge the matching conditions are that the streamfunction and its first derivatives are continuous. At the trailing edge the streamfunction and its first derivatives, as well as the pressure, density and tangential velocity, are continuous. This is the Kutta condition.

The iteration process is based on an elliptic partial differential equation. In Part VI it is also shown that the complete nonlinear differential equation, which includes the compressibility term, becomes "hyperbolic" when the meridional velocity or, if

blades are present, the relative velocity, becomes supersonic. The boundary values for this case are quite complicated and solutions may not be unique. In all the examples solved here the governing velocity is subsonic, although the total velocity may be transonic, as in Part X.

F) The Limiting Flow Far Downstream of the Blades

If at a finite distance downstream of the blades the channel boundaries become concentric cylindrical surfaces and if these surfaces extend unchanged downstream to infinity, general qualitative statements can be made concerning the limiting flow at infinity. It is supposed that the total energy of the fluid is uniform and that the vorticity and tangential velocity are zero on the upstream (inlet) boundary. The energy of the fluid is changed and vorticity is created by the action of an infinite system of blades rotating at a constant angular velocity $\underline{\omega}$. Nothing is said about the channel boundaries upstream of the cylindrical region or about the shape of the blades.

It is clear that in such a region the flow is independent of the axial coordinate and the radial velocity is zero. Then radial equilibrium of the centrifugal force and the pressure gradient force requires that (from Equation (20a)):

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{v^2}{r} \qquad (80)$$

The vorticity, being generated by rotating blades, is parallel to the velocity relative to the blades. Therefore from Equation (42):

$$\frac{dw^2}{dr} = 2\,\frac{\omega r - v}{r}\;\frac{drv}{dr} \tag{81}$$

From Equations (80) and (81) and the isentropy condition, Equation (19), the following relations are derived:

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{r}\,\frac{v^2}{a^2} \tag{82a}$$

$$\frac{da^2}{dr} = (\gamma-1)\,\frac{v^2}{r} \tag{82b}$$

$$\frac{dM_*^2}{dr} = -\frac{2}{r}\,\frac{v^2}{a_*^2}\left(1 - \frac{\omega r}{v^2}\,\frac{d\,rv}{dr}\right) \tag{82c}$$

$$\frac{dM^2}{dr} = -\frac{2}{r}\,\frac{v^2}{a^2}\left(1 + \frac{\gamma-1}{2}M^2 - \frac{\omega r}{v^2}\,\frac{drv}{dr}\right) \tag{82d}$$

The energy equation, Equation (72), can be written as

$$1 + \frac{\gamma-1}{2}M^2 = \frac{a_0^{\,2}}{a^2} + \frac{\gamma-1}{2}\;\frac{2\,\omega r v}{a^2}$$

Using this,

$$\frac{dM^2}{dr} = -\frac{2}{r}\,\frac{v^2}{a^2}\,\frac{a_0^{\,2}}{a^2}\left(1 + \frac{\gamma-1}{2}\;\frac{2\,\omega r v}{a_0^{\,2}}\right) \tag{82e}$$

If rv is a known function of r or of ψ in this region the flow can be completely determined by the above equations. It can be seen directly from Equations (80), (82a), and (82b) that the pressure, the density, and the local speed of sound are minimum on the inner boundary and increase monotonically to a maximum on the outer

boundary. This general result is completely independent of the blade shape and the channel shape. If the blades are stationary it follows from Equations (82c) and (82e) that the total velocity and the corresponding Mach number are maximum on the inner boundary and decrease monotonically to a minimum on the outer boundary. The axial velocity may have a maximum or minimum anywhere, depending on rv, but if rv is monotonic then the axial velocity is also monotonic.

In order to examine the total mass flow the channel is given the same cross-sectional area far upstream and far downstream. The total energy of the fluid is constant. If the tangential velocity is zero upstream and constant downstream the total mass flow equation is

$$\rho_1 w_1 \left(\frac{r_o^2 - 1}{2} \right) = \int_1^{r_o} \rho_2 w_2 \, r \, dr \qquad (83)$$

where $(\ )_1$ and $(\ )_2$ denote values far upstream and far downstream respectively.

Using the isentropic relation,

$$\frac{\rho}{\rho_o} = \left(1 - \frac{\gamma-1}{\gamma+1} M_*^2 \right)^{\frac{1}{\gamma-1}}$$

and integrals of Equation (82), Equation (83) becomes

$$\left(1 - \frac{\gamma-1}{\gamma+1} M_1^{*2} \right)^{\frac{1}{\gamma-1}} M_1^* = \frac{2}{r_o^2 - 1} \int_1^{r_o} \left[1 - \frac{\gamma-1}{\gamma+1} \left(M_{2i}^2 - \frac{v_2^2}{a_*^2} \ln r^2 \right) \right]^{\frac{1}{\gamma-1}} \left[M_{2i}^{*2} - \frac{v_2^2}{a_*^2} \left(1 + \ln r^2 \right) \right]^{\frac{1}{2}} r \, dr \qquad (84)$$

where $M_{2i}^2 = \left(\frac{w}{a_*} \right)_{2i}^2 + \frac{v^2}{a_*^2}$ is the maximum which occurs at the inner radius, taken here as unity.

The solution of Equation (84) is shown in Figure 43 where

$\left(\frac{w}{a_*}\right)^2_{2i}$ is plotted as a function of M_1^{*2} for several values of $\frac{v^2}{a_*^2}$. These results are expressed in terms of local Mach numbers in Figure 44, where the axial Mach number $\left(\frac{w}{a}\right)^2_{2i}$, is plotted as a function of the inlet Mach number M_1^2 for several values of $\frac{v^2}{a^2_{2i}}$. It is seen that for a given inlet Mach number, M_1^2, and a given deflection, $\left(\frac{v}{a}\right)^2_{2i}$, there may be two solutions for the maximum axial Mach number downstream, but if $\left(\frac{v^2}{a^2}\right)_{2i}$ is above a certain critical value there will be no solution. When two solutions exist one value of the axial velocity may be subsonic and the other supersonic, or both values may be supersonic. It must be understood that $\left(\frac{w}{a}\right)_{2i}$ and $\left(\frac{w}{a_*}\right)_{2i}$ are maximum values on the downstream boundary and that when these are supersonic the minimum velocities, and even the mean velocities, may still be subsonic.

These results depend only on the limiting flows far upstream and far downstream and are independent of the intermediate flow. It is assumed, however, that the intermediate channel and the blades are such that an isentropic transition from the upstream flow to the downstream flow is possible. Nevertheless, this analysis shows that for a given inlet Mach number there is a maximum tangential velocity that may be generated by stationary blades. No isentropic solution exists for tangential velocities exceeding this maximum. Similar results would be obtained for distributions of tangential velocity other than constant, and for rotating blades.

VI. PROPERTIES OF THE TRANSONIC FLOW OF A PERFECT FLUID UNDER THE ACTION OF A PSEUDO-CONSERVATIVE FORCE FIELD

A) The Characteristics of the Problem--The Governing Velocity

The applicable equations upstream and downstream of the force field are shown in Equations (19), (20), and (21) with $\lambda = 0$. Adding to these the four differentials for u, rv, w, ln ρ, and using Equation (19) to eliminate the pressure terms, eight algebraic equations for the eight partial derivatives are obtained:

$$
\begin{aligned}
&u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} + a^2\frac{1}{\rho}\frac{\partial \rho}{\partial r} = \frac{v^2}{r}\\
&u\frac{\partial rv}{\partial r} + w\frac{\partial rv}{\partial z} = 0\\
&u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} + a^2\frac{1}{\rho}\frac{\partial \rho}{\partial z} = 0\\
&\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + u\frac{1}{\rho}\frac{\partial \rho}{\partial r} + w\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{u}{r}\\
&dr\frac{\partial u}{\partial r} + dz\frac{\partial u}{\partial z} = du\\
&dr\frac{\partial rv}{\partial r} + dz\frac{\partial rv}{\partial z} = d\,rv\\
&dr\frac{\partial w}{\partial r} + dz\frac{\partial w}{\partial z} = dw\\
&dr\frac{1}{\rho}\frac{\partial \rho}{\partial r} + dz\frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{1}{\rho}d\rho
\end{aligned}
\qquad (85)
$$

The differential equations for the characteristics are obtained when the determinant of the coefficients is set equal to zero:

$$\begin{vmatrix} u & w & 0 & 0 & 0 & 0 & a^2 & 0 \\ 0 & 0 & u & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u & w & 0 & a^2 \\ 1 & 0 & 0 & 0 & 0 & 1 & u & w \\ dr & dz & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & dr & dz & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dr & dz & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & dr & dz \end{vmatrix} = 0 \qquad (86)$$

Expanded, this equation is:

$$w^2(w^2-a^2)dr^4 + 2uw(2w^2-a^2)dr^3dz + [6u^2w^2-(u^2+w^2)a^2]dr^2dz^2$$
$$+ 2uw(2u^2-a^2)drdz^3 + u^2(u^2-a^2)dr^4 = 0$$

which on factoring yields:

$$\left(\frac{dr}{dz} - \frac{u}{w}\right)^2 = 0 \qquad (87)$$

$$\frac{dr}{dz} = \frac{uw \pm a\sqrt{u^2+w^2-a^2}}{w^2-a^2} \qquad (88)$$

Equations (87) and (88) are the differential equations for the physical characteristics of the problem. These equations represent, respectively, the streamsurfaces and the "meridional Mach surfaces". Thus the character of the flow upstream and downstream of the blades depends on the meridional velocity, not the total velocity. That is, the problem is "elliptic" with respect to the variables u and w if the meridional velocity is subsonic, and is "hyperbolic" if it is supersonic.

The velocity which determines the character of the problem is called the "governing velocity", thus the governing velocity upstream and downstream of the blades is the meridional velocity.

In the region of the blades the applicable equations are those given above but with $\lambda \neq 0$. In addition Equation (25), which states that the relative velocity is tangent to the blade surfaces, must be included. Equation (19) can again be used to eliminate the pressure and Equation (20b) can be used to eliminate λ. Because of the constraint imposed by the blade surfaces (Equation (25)) it is also possible to eliminate the angular momentum rv. The blade surface function, f(r,z) is prescribed. Then, including the differentials for u, w, and ln ρ, there are six linear algebraic equations for the six partial derivatives:

$$\begin{aligned}
(1+R^2)\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)+RZ\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)+a^2\frac{1}{\rho}\frac{\partial \rho}{\partial r} &= \frac{v^2-RG}{r}\\
RZ\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)+(1+Z^2)\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)+a^2\frac{1}{\rho}\frac{\partial \rho}{\partial z} &= -\frac{1}{r}ZG\\
\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}+u\frac{1}{\rho}\frac{\partial \rho}{\partial r}+w\frac{1}{\rho}\frac{\partial \rho}{\partial z} &= -\frac{u}{r}\\
dr\frac{\partial u}{\partial r}+dz\frac{\partial u}{\partial z} &= du\\
dr\frac{\partial w}{\partial r}+dz\frac{\partial w}{\partial z} &= dw\\
dr\frac{1}{\rho}\frac{\partial \rho}{\partial r}+dz\frac{1}{\rho}\frac{\partial \rho}{\partial z} &= \frac{1}{\rho}d\rho
\end{aligned} \tag{39}$$

where

$$G = 2\omega r u + u^2\frac{\partial}{\partial r}rR + uw\left(\frac{\partial}{\partial r}rZ+\frac{\partial}{\partial z}rR\right)+w^2\frac{\partial}{\partial z}rZ$$

$$v^2 = (\omega r + Ru + Zw)^2$$

Setting the determinant of the coefficients equal to zero the following equation is obtained:

$$w\left\{(1+R^2+Z^2)w^2-(1+R^2)a^2\right\}dr^3-\left\{3(1+R^2+Z^2)uw^2-\left[(1+R^2)u-2RZw\right]a^2\right\}dr^2dz$$
$$+\left\{3(1+R^2+Z^2)u^2w-\left[(1+Z^2)w-2RZu\right]a^2\right\}dr\,dz^2$$
$$-u\left\{(1+R^2+Z^2)u^2-(1+Z^2)a^2\right\}dz^3=0 \qquad (90)$$

On factoring this yields:

$$w\,dr-u\,dz=0\ , \qquad \text{and}$$

$$\left(\frac{1+R^2+Z^2}{1+R^2}w^2-a^2\right)(1+R^2)dr^2-2\left(\frac{1+R^2+Z^2}{\sqrt{1+R^2}\sqrt{1+Z^2}}uw+\frac{RZa^2}{\sqrt{1+R^2}\sqrt{1+Z^2}}\right)\sqrt{1+R^2}\sqrt{1+Z^2}\,dr\,dz$$
$$+\left(\frac{1+R^2+Z^2}{1+Z^2}u^2-a^2\right)(1+Z^2)dz^2=0$$

The differential equations for the physical characteristics are therefore:

$$\frac{dr}{dz}=\frac{u}{w} \qquad (91)$$

$$\frac{dr}{dz}=\frac{(1+R^2+Z^2)uw+RZa^2\pm a\sqrt{(1+R^2+Z^2)\left[u^2+w^2+(Ru+Zw)^2-a^2\right]}}{(1+R^2+Z^2)w^2-(1+R^2)a^2} \qquad (92)$$

where $Ru+Zw$ is the relative tangential velocity, $(v-\omega r)$.

The characteristics of the flow in the region of the blades are the streamsurfaces (Equation 91) and the surfaces given by Equation (92). Whether or not the characteristics given by Equation (92) are real depends on the total *relative* velocity, not on the meridional component of the velocity as in the upstream and downstream regions. The governing velocity in the region of the blades is therefore the

velocity relative to the blades.

The form of Equation (92) suggests the following transformation:

$$\bar{u} = \frac{\sqrt{1+R^2+Z^2}}{\sqrt{1+Z^2}}\, u \quad ; \quad \bar{w} = \frac{\sqrt{1+R^2+Z^2}}{\sqrt{1+R^2}}\, w$$

$$d\bar{r} = \sqrt{1+R^2}\, dr \quad ; \quad d\bar{z} = \sqrt{1+Z^2}\, dz \tag{93}$$

After this transformation Equations (91) and (92) become:

$$\frac{d\bar{r}}{d\bar{z}} = \frac{\bar{u}}{\bar{w}}$$

$$\frac{d\bar{r}}{d\bar{z}} = \frac{\bar{u}\bar{w} + \dfrac{2RZa^2}{\sqrt{1+R^2}\sqrt{1+Z^2}} \pm a\sqrt{\bar{u}^2 + \bar{w}^2 + \dfrac{2RZ(1+R^2+Z^2)}{(1+R^2)(1+Z^2)}\bar{u}\bar{w} - \dfrac{1+R^2+Z^2}{(1+R^2)(1+Z^2)}a^2}}{\bar{w}^2 - a^2} \tag{94}$$

This is no great simplification except when R or Z is zero everywhere. For radial blade R is zero, the transformation is:

$$\bar{u} = u \quad ; \quad \bar{w} = \sqrt{1+Z^2}\, w$$

$$d\bar{r} = dr \quad ; \quad d\bar{z} = \sqrt{1+Z^2}\, dz \tag{95}$$

and the equations for the characteristics, in terms of the transformed variables are:

$$\frac{d\bar{r}}{d\bar{z}} = \frac{\bar{u}}{\bar{w}}$$

$$\frac{d\bar{r}}{d\bar{z}} = \frac{\bar{u}\bar{w} \pm a\sqrt{\bar{u}^2 + \bar{w}^2 - a^2}}{\bar{w}^2 - a^2} \tag{96}$$

Thus the characteristics for the transformed problem are the same as for two-dimensional flow with velocities $\bar{u}$ and $\bar{w}$ in the $\bar{r}$, $\bar{z}$ space.

B) The Complete Non-linear Forms of the Equation

With uniform inlet conditions the equation for the stream-function for compressible flow downstream of a system of blades

rotating at angular velocity ω is (from Equation (65)):

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = \psi_n (\ln \rho)_n - \left(\frac{\rho}{\rho_*}\right)^2 \left[\frac{rv}{a_*} - \frac{\omega r^2}{a_*}\right] \frac{d}{d\psi} \frac{rv}{a_*} \tag{97}$$

The density derivatives can be written using Equation (78) as

$$\begin{aligned} \frac{1}{\rho}\frac{\partial \rho}{\partial r} &= \frac{\frac{1}{r}(\psi_r^2 + \psi_z^2) - \psi_r \psi_{rr} - \psi_z \psi_{zr} - r^2 \left(\frac{\rho}{\rho_*}\right)^2 \frac{\partial}{\partial r} \frac{v^2}{a_*^2}}{r^2 \left(\frac{\rho}{\rho_*}\right)^2 \left(\frac{a}{a_*}\right)^2 - (\psi_r^2 + \psi_z^2)} \\ \frac{1}{\rho}\frac{\partial \rho}{\partial z} &= \frac{-\psi_r \psi_{rz} - \psi_z \psi_{zz} - r^2 \left(\frac{\rho}{\rho_*}\right)^2 \frac{\partial}{\partial z} \left(\frac{v}{a_*}\right)^2}{r^2 \left(\frac{\rho}{\rho_*}\right)^2 \left(\frac{a}{a_*}\right)^2 - (\psi_r^2 + \psi_z^2)} \end{aligned} \tag{98}$$

Then, eliminating the density derivatives from Equation (97) and introducing the velocity components, the complete non-linear form of the equation for the downstream region is obtained.

$$\begin{aligned} &\left(1 - \frac{u^2}{a^2}\right)\psi_{rr} + 2\frac{uw}{a^2}\psi_{rz} + \left(1 - \frac{w^2}{a^2}\right)\psi_{zz} - \left(1 - \frac{\rho_* a_*^2}{\rho a^2} \frac{1}{r} \frac{\partial}{\partial r} \frac{v^2}{a_*^2}\right) \frac{1}{r}\psi_r \\ &\quad + \frac{\rho_* a_*^2}{\rho a^2} \frac{1}{r^2} \frac{\partial}{\partial z} \frac{v^2}{a_*^2} \psi_z = \left\{ - \frac{\rho^2}{\rho_*^2} \left[\frac{rv}{a_*} - \frac{\omega r^2}{a_*}\right] \frac{d}{d\psi} \frac{rv}{a_*} \right\} \left\{ 1 - \frac{u^2 + w^2}{a^2} \right\} \end{aligned} \tag{99}$$

The non-homogeneous part, "the right side', represents the effect of rotationality. The terms containing the derivatives of $\frac{v^2}{a_*^2}$ are connected with the fact that part of the total energy occurs as kinetic energy due to the tangential velocity. It is clear that the effect of rotationality becomes zero if the meridional velocity is sonic, and that the rotationality effect on one side of the sonic line is exactly opposite the effect on the other side.

The equation for the streamfunction for compressible flow in a region of radial blades is (from Equation (70)):

$$(1 + Z^2)\psi_{rr} - (1 - Z^2)\frac{1}{r}\psi_r + \psi_{zz} - (1 + Z^2)\frac{1}{\rho}\frac{\partial \rho}{\partial r}\psi_r - \frac{1}{\rho}\frac{\partial \rho}{\partial z}\psi_z = 2\frac{\omega r}{a_*}\frac{\rho}{\rho_*} Z \tag{100}$$

The density derivatives, from Equation (78), can be written as:

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = \frac{\frac{\omega^2 r}{a^2} - \left(\frac{\rho_*}{r\rho a}\right)^2 \left\{\psi_z \psi_{rz} + (1+Z^2)\psi_r \psi_{rr} + \psi_z^2 Z \frac{\partial Z}{\partial r}\right\} + \frac{1}{r}\,\frac{u^2+(1+Z^2)w^2}{a^2}}{1 - \frac{u^2+(1+Z^2)w^2}{a^2}}$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{-\left(\frac{\rho_*}{r\rho a}\right)^2 \left\{\psi_z \psi_{zz} + (1+Z^2)\psi_r \psi_{rz} + \psi_r^2 Z \frac{\partial Z}{\partial z}\right\}}{1 - \frac{u^2+(1+Z^2)w^2}{a^2}} \qquad (101)$$

The complete non-linear equation for the region of the blades is obtained by eliminating the density derivatives from Equation (100) and introducing the velocity components in the coefficients:

$$\left(1-\frac{u^2}{a^2}\right)(1+Z^2)\psi_{rr} + \frac{2uw}{a^2}(1+Z^2)\psi_{rz} + \left[1-(1+Z^2)\frac{w^2}{a^2}\right]\psi_{zz}$$
$$-\left\{(1-Z^2)+(1+Z^2)\frac{\omega^2 r^2}{a^2} + \left[2\frac{u^2}{a^2}+(1+Z^2)w^2\right] r Z \frac{\partial Z}{\partial r}\right\}\frac{1}{r}\psi_r$$
$$+\left(\frac{w^2}{a^2} Z \frac{\partial Z}{\partial z}\right)\psi_z \quad = 2\frac{\omega r}{a_*}\frac{\rho}{\rho_*} Z\left[1-\frac{u^2+(1+Z^2)w^2}{a^2}\right] \qquad (102)$$

C) The Cushioning Action between Subsonic and Supersonic Regions

The complete non-linear equations (Equations (99) and (102)) are obviously quite complicated. It is believed, however, that the transformation suggested (Equation (95)) would lead to some simplification of Equation (102). Unfortunately, time has not permitted a more thorough investigation of the possibilities of this transformation.

Examination of Equations (99) and (102) and the characteristic equations, Equations (88) and (92), reveals that the non-homogeneous part, the right side of the equation, changes sign whenever

the character of the flow changes from "elliptic" to "hyperbolic". Furthermore, for flows of practical interest the vorticity term

$$\left\{ \left(\frac{\rho}{\rho_*}\right)^2 \left[\frac{rv}{a_*} - \frac{\omega r^2}{a_*}\right] \frac{d}{d\psi} \frac{rv}{a_*} \right\}, \quad \text{or} \quad 2\frac{\omega r}{a_*}\frac{\rho}{\rho_*} Z$$

will have the same sign throughout the flow region. The right side of the equation can be regarded as a forcing function acting to displace the streamlines. In regions where the flow is entirely subsonic or entirely supersonic the forcing function will have one sign throughout the region. But if the flow is transonic, i.e. part subsonic and part supersonic, the forcing function has one sign in the subsonic part and the opposite sign in the supersonic part. Therefore the deflection of the streamlines, brought about by vorticity, will be less if the governing velocity is transonic than if it is entirely subsonic or entirely supersonic. Consequently the meridional flow will be "smoother", for the vorticity in one region tends to counteract that of the other region. It is believed that this mutual cushioning effect is the explanation of the phenomenal efficiences observed in compressors in which the relative velocities in a region near the tip are supersonic.

The fact that the forcing function, the non-homogeneous part of the equation, contains a factor of the form $(1-M_g^2)$, where M_g is the "governing" Mach number, leads to the following general conclusions:

1. The deflection of the streamsurfaces induced by a given strength of vorticity at a certain point in the flow is zero when the governing velocity is sonic at this point, has one sense when the velocity is subsonic, and the opposite sense when it is supersonic.

2. The deflection of the streamlines brought about by vorticity in a region is less when the governing velocity is transonic in the region than if the velocity is entirely subsonic or entirely supersonic.

These conclusions are confirmed in all of the examples of compressible flow which follow. In fact, it seems that even stronger statements should be made. A comparison of the deflection of the streamsurfaces brought about by vorticity indicates that the deflections increase from zero to a maximum and then decrease as the vorticity (the strength of the actuator) is increased from zero. The governing velocity is subsonic in both examples.

VII. THE FORMULATION OF THE FINITE DIFFERENCE PROBLEM

The finite difference problem differs from the differential equation problem in two ways: in the finite difference problem, the boundary conditions are prescribed at a finite number of points on the boundary and the desired function is to be determined at a finite number of points within the boundary; the differential equation is essentially replaced by a number of simultaneous approximate difference equations which are themselves only approximately solved. However, the set of points at which the desired function is sought can be made very dense, and the system of difference equations can be solved as accurately as desired. Therefore, barring unaccounted for singularities, any degree of agreement between the two solutions may be obtained. The two problems are nevertheless quite distinct. The finite difference problem is solved by Southwell's relaxation method (Reference 8), which is defined as "a systematic sequence of localized changes of the wanted function that steadily brings the 'residuals' toward their desired value."

The fact that the physical aspects of the problem are always evident and that the status of the solution is always apparent is a great advantage of the relaxation method. This is a great aid in solving the non-linear turbomachine problem, for the action of the vorticity and the compressibility is apparent as a "force" which causes the streamlines to deform from their incompressible, irro-

tational positions.

The power of the relaxation method is indicated in the following quotation from Southwell (Reference 6, p. 3):

> "(The) use of finite differences is not new, nor (is the) evaluation of the wanted function at nodal points of a regular net. But in concentrating attention on the data, and in recognizing that these are never exact, they subordinate mathematical to physical aspects in a way that can alter drastically the course of a theoretical research. Discarding orthodox for relaxation methods, an investigator finds his outlook quite transformed: full scope remains for ingenuity and special artifice, but any problem that can be formulated can be solved."

It is not clear how solutions which are unstable can always be found by the method of finite differences. Intuitively it seems that if the configuration is unstable the disturbances inherent in the relaxation process may be amplified by the relaxation, thus making convergence questionable.

A) The Difference Equations

The most general form of the differential equation for the streamfunction is that shown by Equation (64):

$$A\psi_{rr} + 2B\psi_{rz} + C\psi_{zz} + D = 0 \qquad (103)$$

where A, B, C are functions of r and z, and D is a function of ψ_r, ψ_z, r, and z. As written for each step of the iteration with the non-homogeneous part known, Equation (64) is an elliptic equation, for

$$AC - B^2 = 1 + R^2 + Z^2 > 0,$$

and can therefore always be transformed into the normal form

$$\psi_{xx} + \psi_{yy} + D'(x, y, \psi_r, \psi_z) = 0 \qquad (104)$$

When the cross derivative term ψ_{rz} occurs, as in Equation (64), the difference equation may be written for either the original equation or the transformed normal equation. Because of the added difficulty of matching solutions when one region has been transformed, the difference equations are written here for the original form.

If the streamfunction ψ, which is a solution to a partial differential equation such as Equation (64), is regular in some neighborhood of the point (r_o, z_o), it may be represented in this neighborhood by the Taylor's expansion:

$$\begin{aligned} \psi(r,z) = {} & \psi_o + \psi_r(r-r_o) + \psi_z(z-z_o) \\ & + \frac{1}{2!}\left[\psi_{rr}(r-r_o)^2 + 2\,\psi_{rz}(r-r_o)(z-z_o) + \psi_{zz}(z-z_o)^2\right] \\ & + ---- \end{aligned} \qquad (105)$$

where ψ_o is the value of ψ at (r_o, z_o) and all the derivatives are evaluated at (r_o, z_o). Therefore if (r_o, z_o) is one point of a square lattice with an interval between points of δ,

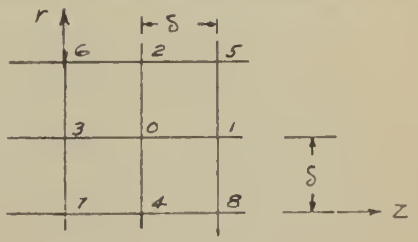

Figure 6
The Square Lattice or Net

the value of ψ at the surrounding points can be expressed by means of Equations (105). If fourth and higher powers of δ are neglected the result is a system of simultaneous linear algebraic equations which can be solved for the derivatives to give:

$$\begin{aligned} \psi_r &\approx \frac{\psi_2 - \psi_4}{2\delta}; \qquad \psi_z \approx \frac{\psi_1 - \psi_3}{2\delta} \\ \psi_{rr} &\approx \frac{\psi_2 + \psi_4 - 2\psi_0}{\delta^2}; \quad \psi_{zz} \approx \frac{\psi_1 + \psi_3 - 2\psi_0}{\delta^2} \\ \psi_{rz} &\approx \frac{\psi_5 - \psi_6 + \psi_7 - \psi_8}{4\delta^2} \end{aligned} \tag{106}$$

On eliminating the first order derivatives it follows that:

$$\begin{aligned} \psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0 &\approx (\psi_{rr} + \psi_{zz})\delta^2 \\ \psi_5 + \psi_6 + \psi_7 + \psi_8 - 4\psi_0 &\approx (\psi_{rr} + \psi_{zz})\delta^2 \end{aligned} \tag{107}$$

The difference equation is obtained when Equation (106) is substituted into the differential equation. Thus for Equation (64), which is

$$(1+Z^2)\psi_{rr} - 2RZ\psi_{rz} + (1+R^2)\psi_{zz} + (ZZ_r - RZ_z - \tfrac{1}{r})\psi_r - (ZR_r - RR_z)\psi_z = \mathcal{Q} \tag{108}$$

the difference equation is:

$$\begin{aligned} &\left[(1+R^2) - (ZR_r - RR_z)\tfrac{\delta}{2}\right]\psi_1 + \left[(1+Z^2) + (ZZ_r - RZ_z - \tfrac{1}{r})\tfrac{\delta}{2}\right]\psi_2 \\ &+ \left[(1+R^2) + (ZR_r - RR_z)\tfrac{\delta}{2}\right]\psi_3 + \left[(1+Z^2) - (ZZ_r - RZ_z - \tfrac{1}{r})\tfrac{\delta}{2}\right]\psi_4 \\ &- \tfrac{1}{2}RZ(\psi_5 - \psi_6 + \psi_7 - \psi_8) - 4\left[1 + \tfrac{1}{2}(R^2 + Z^2)\right]\psi_0 = \mathcal{Q}\delta^2 \end{aligned} \tag{109}$$

where

$$\begin{aligned} \mathcal{Q} = \frac{2\omega r}{a_*}\frac{\rho}{\rho_*}Z + \left[(1+Z^2)\psi_r - RZ\psi_z\right](\ln\rho)_r + \left[(1+R^2)\psi_z - RZ\psi_r\right](\ln\rho)_z \\ + \left(\frac{\rho}{\rho_*}\right)^2 r^2 \frac{d}{d\psi}\left(\frac{E_1}{a_*^2} - \frac{\omega r v_1}{a_*^2}\right) \end{aligned}$$

When the blade shape function f(r, z) is prescribed all the coefficients can be determined. Equation (109) is then the difference equation applicable in the region of the blades. The residual, $\mathcal{R}\delta^2$, is a complicated function of position which is estimated for each step of the iteration by means of the solution of the previous step.

In the same way the difference equation upstream of the blades is, from Equation (63):

$$\psi_1 + \left(1-\frac{\delta}{2r}\right)\psi_2 + \psi_3 + \left(1+\frac{\delta}{2r}\right)\psi_4 - 4\psi_0 = \mathcal{R}\delta^2 \qquad (110)$$

where

$$\mathcal{R} = \psi_n (\ln \rho)_n + \left(\frac{\rho}{\rho_*}\right)^2 \left[r^2 \frac{d}{d\psi} \frac{E_1}{a_*^2} - \frac{1}{2} \frac{d}{d\psi} \left(\frac{rv}{a_*}\right)_1^2 \right]$$

The difference equation for the downstream region is the same as Equation (110), but the residual, from Equation (65), is given by

$$\begin{aligned} \mathcal{R} = \psi_n (\ln \rho)_n &- \left(\frac{\rho}{\rho_*}\right)^2 \left[\left(\frac{rv}{a_*}\right)_t - \frac{\omega r^2}{a_*} \right] \frac{d}{d\psi} \left(\frac{rv}{a_*}\right)_t \\ &+ \left(\frac{\rho}{\rho_*}\right)^2 \left[r^2 \frac{d}{d\psi} \frac{E_1}{a_*^2} - \frac{\omega r^2}{a_*} \frac{d}{d\psi} \left(\frac{rv}{a_*}\right)_1 \right] \end{aligned} \qquad (111)$$

The difference equation for each of the three regions is shown in Equations (109), (110), and (111). A general form might be written as

$$D_i(\psi) = \delta^2 \mathcal{R} \qquad (112)$$

Here $D_i(\psi)$ represents the left side of one of the difference equations written for the i^th^ point of the net and $\delta^2\mathcal{R}$ is the residual there. $D_i(\ )$ is the "difference operator", and is itself a function

of position. That is, the coefficients of the left side of the difference equations have different values at each net point and these values depend on the prescribed shape of the blades.

If the blades are prescribed as radial, the expression for D_i is more simple than in the general form (Equation (109)). For radial blades

$$\begin{aligned} D_i(\psi) = {} & \psi_1 + \left[(1+Z^2) - (1-Z^2)\frac{\delta}{2r}\right]\psi_2 + \psi_3 \\ & + \left[(1+Z)^2 + (1-Z^2)\frac{\delta}{2r}\right]\psi_4 - 4\left[1 + \frac{1}{2}Z^2\right]\psi_0 \end{aligned} \tag{113}$$

in the region of the blades. The $\mathcal{R}_i$ are correspondingly more simple for radial blades. However, the complexity of $\mathcal{R}$ is primarily due to rotationality and compressibility. For example, if the fluid is incompressible and the inlet conditions are uniform the right side of Equation (64) becomes simply

$$\mathcal{R}_i = 2\omega r Z \ , \tag{114}$$

regardless of the blade shape prescribed.

Specialized forms of the difference equations will appear in the various examples, Parts VIII, IX, and X. The general form of the difference operator is written as the sum:

$$D_i(\psi) = \sum_j a_{ij}\,\psi_{ij} \tag{115}$$

The subscript i denotes the point of the net for which the operation is performed and the subscript j denotes the values of the streamfunction which enter into the difference equation. Thus for the simplest difference operator, from Equation (110):

$$\begin{aligned} D_i(\psi) &= \psi_{i1} + \left(1-\frac{\delta}{2r_i}\right)\psi_{i2} + \psi_{i3} + \left(1+\frac{\delta}{2r_i}\right)\psi_{i4} - 4\psi_{i0} \\ a_{i1} &= a_{i3} = 1 \quad ; \quad a_{i0} = -4 \\ a_{i2} &= 1-\frac{\delta}{2r_i} \quad ; \quad a_{i4} = 1+\frac{\delta}{2r_i} \end{aligned} \tag{116}$$

The a_{ij} are called "influence coefficients" in that they indicate the change in $\delta^2 R_i$, corresponding to a unit change in ψ_{ij} ; or $\frac{1}{a_{ij}}$ represents the change in ψ_{ij} due to a unit change in the "forcing function" $\delta^2 R_i$ when all other ψ_{ij} are held constant. The a_{ij} from Equation (116) are shown for some point i:

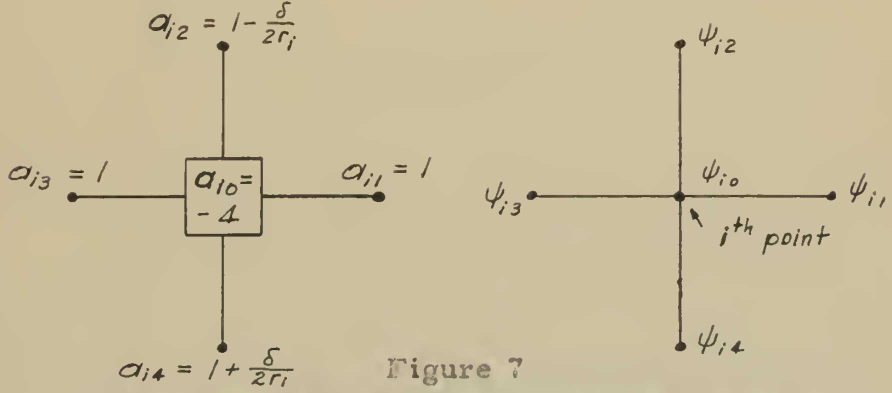

Figure 7
The Nomenclature of the Difference Operator

The difference equations must be satisfied at every point in the net. This is accomplished by Southwell's relaxation technique, whereby the streamfunction is modified until the desired residuals are obtained.

The solution of a finite difference problem is generally not the same as the solution of the corresponding differential equation problem. The amount of the discrepancy depends on the nature

of the function involved and on the size of the net used. Therefore it is only necessary to obtain an accuracy in the solution of the difference problem commensurate with the discrepancy already involved.

B) The Iteration Process

It is not practical to attempt to formulate a general set of rules which will govern the iteration process in all conceivable problems. It is, however, appropriate to discuss generally the two separate effects--compressibility and vorticity. For the initial step of the iteration it is necessary to estimate the right sides of the equations. Consider, for example, the right side of Equation (65):

$$\psi_n (\ln \rho)_n - \frac{\rho^2}{\rho_*^2}\left[\left\{\left(\frac{rV}{a_*}\right)_t - \frac{\omega r^2}{a_*}\right\} \frac{d}{d\psi}\left(\frac{rV}{a_*}\right)_t\right]$$

The manner in which rv(ψ) may be estimated has already been discussed. In the examples solved the first estimates of rv(ψ), so obtained, were accurate to within three percent of the final values (Figures 21 and 23). The complete effect of rotationality, represented by the term in the bracket above, can likewise be expressed as a function of ψ and r with comparable accuracy.

In the relaxation process it is possible to improve the desired residual each time the streamfunction is improved, so that the relaxation and the iteration are actually performed simultaneously.

Periodically $rv(\psi)$ must be recalculated and a new, more accurate function of ψ and r obtained.

The compressibility effect cannot be handled so easily. If the Mach number is low and the rotationality effect is strong, as in the examples of Part IX, the first term, $\psi_n (\ln \rho)_n$, is much smaller than the last term and can be disregarded for the first estimation. In this case the effect of compressibility is not completely neglected, but appears in the $\left(\frac{\rho}{\rho_x}\right)^2$ factor multiplying the rotationality term. It is true that under these circumstances the main effect of compressibility is its "influence" on the rotationality. For large subsonic Mach numbers this approach is not possible. The first term may be of the same order as the second. In fact, in the examples of Part X, where a constant tangential velocity was prescribed at the downstream boundary, the two terms were exactly equal at the point on the downstream boundary where the axial velocity became sonic. This can be seen by writing Equation (65), with $\omega = 0$ and $\psi_{zz} = 0$, as

$$r \frac{\partial}{\partial r}\left(\frac{1}{r}\psi_r\right) - r \frac{\partial}{\partial r}(\rho w) \sim R$$

It is well known that if the energy is uniform the local mass flow is a maximum when w is sonic, hence R is zero when w is sonic. Another viewpoint is to regard the residuals as forcing functions which act to displace the streamlines. Since the axial velocity can

only be sonic at a "throat", the forces must act toward the sonic line so as to contract the streamlines there. The residuals, having a different sense on each side of the sonic line and being "continuous", must therefore be zero on the sonic line. When v is not constant or when the energy is non-uniform, ρw will not necessarily be maximum when w is sonic. The exact circumstances under which the residuals change sign were derived in Part VI, where it was shown that the residuals are zero on the line separating the "hyperbolic" and the "elliptic" regions, or, in other words, where the "governing velocity" is sonic.

It is extremely important to use every means available to make the first estimate of the compressibility effect as good as possible. The rapidity of convergence depends on the judgement used in the first estimation and each successive approximation. No general rules can be stated. In regions where the meridional velocity is nearly sonic, the streamsurfaces obtained during the iteration may be such that the channel between adjacent surfaces is "choked". When this happens the mass flow ρq_s falls to the right of the density curve in Figure 42 and the density is imaginary. If it is not possible to adjust the streamlines so as to have real densities, then it may be that the governing velocity is supersonic and the "elliptic" form of the equation is not applicable, or even that no isentropic solution exists. If there is reason to believe isentropic solutions exist,

the "complete non-linear" equations derived in the appendix may be used for the transonic problem. Convergence is always slow when the governing velocity is nearly sonic, for the ρ vs ρq_s curves have nearly vertical tangents then.

VIII. AN EXAMPLE OF INCOMPRESSIBLE FLOW IN A MIXED FLOW COMPRESSOR WITH BLADE SHAPE PRESCRIBED.

This analysis of the flow in a mixed-flow compressor with blade shape prescribed is presented as an example of the three-dimensional motion of an incompressible fluid under the action of a system of rotating blades. First a solution is obtained for the irrotational flow through the channel with no blades present. This serves as a basis of comparison for determining the additional velocities induced by the blades, and is also useful in making the first approximation to the rotational flow when blades are present.

The particular channel chosen is shown in Figure 8. The curved part of the channel boundary can be expressed analytically as

$$\frac{r - r_1}{r_2 - r_1} = \frac{z}{L} - \frac{\sin 2\pi \frac{z}{L}}{2\pi} \tag{117}$$

where r_1 is the initial radius, r_2 is the final radius, and L is the length of the curved portion. The coordinates of the channel boundaries are given in Table I.

In this example the total energy is uniform and the vorticity is zero at the inlet, station 0. More precisely, the tangential velocity is zero, the meridional velocity is axial, and the pressure is constant on the upstream boundary. In the blades and downstream of the blades the total energy is non-uniform and the flow is rotational. The downstream boundary conditions are essentially those

that might be expected if the channel extended downstream to infinity: the flow is independent of the axial coordinate. Thus the radial velocity is zero and the pressure gradient forces and the centrifugal forces are in balance on the downstream boundary.

A) Irrotational Flow with no Blades Present

For irrotational flow with no blades present the governing differential equation for the streamfunction (Equation (63)) is simply:

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = 0 \qquad (118)$$

and from Equation (110) the corresponding difference equation is:

$$\psi_1 + \left(1 - \frac{\delta}{2r}\right)\psi_2 + \psi_3 + \left(1 + \frac{\delta}{2r}\right)\psi_4 - 4\psi_0 = 0 \qquad (119)$$

The boundary conditions are: ψ = constant on the hub and shroud contours, $-\frac{1}{r}\psi_r = w = 1$ on the upstream boundary, and $\frac{1}{r}\psi_z = u = 0$ on the downstream boundary. Upstream ψ_r is taken along the boundary, so that ψ can be determined by simple integration and prescribed directly. Downstream the normal derivative is prescribed and ψ cannot be determined by integration along the boundary. The boundary value problem is therefore of the mixed type. However, since the flow far downstream is entirely independent of the axial coordinate all axial derivatives vanish and the differential equation (Equation 118) can be integrated easily. By this means ψ itself can be prescribed over all of the boundary. The mathematical problem for the irrotational flow is therefore relatively simple and consists of satisfying simultaneously the difference equations for all points

in the net.

The flow region is divided into a net or lattice and influence coefficients are calculated for each point. The net, the influence coefficients, and the boundary conditions are shown in Figure 9. The influence coefficients near the boundaries were calculated in the same way as Equations (106) and (108), but with a different δ for the "short legs" of the net.

For a first approximation to ψ the streamsurfaces were assumed to have the same shape as the boundaries. This is equivalent to assuming that the axial velocity is constant on each radial line. The actual residuals R_a for the smooth values of ψ were calculated by means of the difference operator with a high degree of accuracy. These served as basic values of ψ and R_a. The change in ψ to make the residuals as small as possible was then determined by the relaxation process. As a final check residuals were again calculated by the difference operator.

The final values of the streamfunction and the corresponding actual residuals are shown in Figure 10. Upstream of station 12 the smallest residual may be as large as 2 and downstream as large as 5. This error corresponds to an error in the solution of the difference problem of about one part in 2000.

The axial and radial velocities for several representative stations are shown in Figures 11 and 12, respectively. The abscissa

of Figure 11 is the deviation from the mean axial velocity:

$$\Delta \bar{w} = \frac{w - w_m}{w_o}$$

where w_o is the inlet velocity and w_m is the mean axial velocity. The ordinate is the non-dimensional radius:

$$\tilde{r} = \frac{r - r_i}{r_o \ r_i}$$

where r_i is the radius of the inner boundary and r_o is the radius of the outer boundary.

The abscissa of Figure 12 is simply:

$$\tilde{u} = \frac{u}{w_o}$$

Lines of equal pressure are shown in Figure 13 for flow through the channel with no blades present. The decrease in pressure occurring far downstream is due to the decrease of the cross-sectional area of the channel and the corresponding increase of the mean velocity. This should be taken into account when evaluating the pressure increase when the rotating blades are present.

B) Rotational Flow with Blades Present

Rotational flow through the channel of the previous example is considered here. The vorticity is generated by a rotating system of blades which acts on the fluid between stations 7 and 14. Radial blades are prescribed, hence the blade shape function f(r,z) depends only on the axial coordinate z. Based on the irrotational meridional velocities of the previous example the function f(z) was

chosen so that the rate of energy input, i.e. the rate of change of the moment of angular momentum rv, was very small near the leading and trailing edges. The discontinuity in v across the leading edges cannot generally be avoided, however the blades were selected such that the jump in v is positive. The rate of energy input is indicated by the slopes of the curves in Figure 20, where the desired and the actual $\frac{v}{\omega r}$ on the radius r = 2.5 are compared. The blade shape function is chosen to give the desired $\frac{v}{\omega r}$ on the basis of the irrotational meridional velocities. The actual $\frac{v}{\omega r}$ is determined from the final meridional velocities with the prescribed blades present. The shape of the blades is shown in Figure 8 and the blade surface function is given in Table I.

The applicable differential equations for the three regions of flow were derived in Part V-B.

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = 0 \qquad \text{upstream of blades} \qquad (66)$$

$$(1+Z^2)\psi_{rr} - (1-Z^2)\frac{1}{r}\psi_r + \psi_{zz}$$

$$= 2\omega r Z \qquad \text{in region of blades} \qquad (67)$$

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = (rv - \omega r^2)\frac{d\,rv}{d\psi} \qquad \text{downstream of blades} \qquad (68)$$

where rv = rv(ψ) only and is evaluated at the trailing edge by Equation (58), which here becomes:

$$(rv)_t = \omega r_t^2\left[1 + \left(\frac{w}{\omega r}Z\right)_t\right] \qquad (120)$$

The corresponding difference equations in regions where the net interval is δ were given by Equations (110), (111), (113) and (114), which for the three regions become, respectively:

$$\psi_1 + \left(1-\frac{\delta}{2r}\right)\psi_2 + \psi_3 + \left(1+\frac{\delta}{2r}\right)\psi_4 - 4\psi_0 = 0 \qquad (121)$$

$$\psi_1 + \left[\left(1-\frac{\delta}{2r}\right) + \left(1+\frac{\delta}{2r}\right)Z^2\right]\psi_2 + \psi_3 + \left[\left(1+\frac{\delta}{2r}\right) + \left(1-\frac{\delta}{2r}\right)Z^2\right]\psi_4 - 4\left[1+\frac{1}{2}Z^2\right]\psi_0 = 2\omega r Z \delta^2 \qquad (122)$$

$$\psi_1 + \left(1-\frac{\delta}{2r}\right)\psi_2 + \psi_3 + \left(1+\frac{\delta}{2r}\right)\psi_4 - 4\psi_0 = \left(rv-\omega r^2\right)\frac{d\,rv}{d\psi}\delta^2 \qquad (123)$$

where $rv(\psi)$ is evaluated by Equation (120).

The influence coefficients for these equations and for the equations which hold near the boundary where δ is not constant are shown in Figure 14.

The boundary value problem is the same as in Part VIII-A, except that here the streamfunction can not be prescribed exactly on the downstream boundary. Instead new downstream boundary conditions must be found for each approximation to $rv(\psi)$. This is easily done by one-dimensional relaxation on the downstream boundary. The first approximation to the downstream boundary values was based on rv determined from the irrotational flow. For comparison, the initial and final downstream boundary values are:

r	3.25	3.0	2.75	2.5	2.25	2.0
Initial ψ	0	973	1857	2654	3367	4000
Final ψ	0	977	1863	2660	3371	4000

The parameters of the problem, based on the maximum radius of 3.25, are:

Pressure coefficient: $\frac{\omega r v}{\frac{1}{2}(\omega r)^2}$ = 3.55

Flow coefficient: $\frac{w}{\omega r}$ = 0.769

Root-tip ratio: $\frac{r_o}{r_i}$ = 3 upstream

= 1.625 downstream

$\frac{r_o - r_i}{r_i}$ = 2 upstream

= .625 downstream

Blade aspect ratio: 0.232

Static pressure increase: $\left(\frac{\Delta p}{q_o}\right)_{blades} - \left(\frac{\Delta p}{q_o}\right)_{no\ blades}$ = 1.570 at tip,

= .756 at root

The extreme tip-root ratio was chosen so as to amplify the three-dimensional properties of the flow, and blades of very low aspect ratio were prescribed in order to stress the importance of blade geometry.

There was nothing unusual in the analysis. Relaxation and iteration were performed simultaneously, and the function rv(ψ) only had to be determined once after the first estimation. The initial and final values of rv(ψ) are plotted in Figure 21. The final values of the streamfunction and the corresponding desired and actual residuals are shown in Figure 15. The axial and radial velocities are plotted for several stations in Figures 16 and 17, using the same variables as already described. Lines of equal pressure are

shown in Figure 18 and lines of equal total energy in Figure 19. The pressure was calculated by a relation derived from Equation (31):

$$\frac{p-p_o}{\frac{1}{2}\rho w_o^2} = \frac{2\omega r v}{w_o^2} + \left[1 - \frac{u^2+v^2+w^2}{w_o^2}\right] \qquad (124)$$

A comparison of the two solutions, with and without blades, does not lead to any extraordinary or unexpected revelations. There are, however, some interesting points that should be discussed:

1. The general effect of the blades on the meridional flow pattern is to increase the velocities near the shroud and decrease those near the hub. This may be considered as an improvement of the meridional flow, for the most likely place for separation is on the hub near station 13. Decreased velocities on the hub therefore lessen the likelihood of separation.

2. The direction of the streamlines, given by $\frac{u}{w}$, is changed very little by the action of the blades.

3. The lowest static pressure, $\frac{\Delta p}{\frac{1}{2}\rho w_o^2}$ at station 13 on the hub, is increased from -1.000 to -0.166 by the action of the blades.

4. Although relatively large changes of axial velocity occur within the blade region, the downstream equilibrium values

differ very little (4.6 percent) from the axial velocity with no blades present. However, the tangential velocity far downstream is approximately proportional to the radius, and is of the same order as the axial velocity: $\frac{v}{W_0}$ = 1.146 at tip, $\frac{v}{W_0}$ = 0.720 at root. Considerable diffusion would therefore be necessary to recover, as static pressure, all of the kinetic energy added.

5. Considerably more energy was added near the shroud than near the hub. This is unavoidable for radial blades, or, in fact, for any "practical" blades of low aspect ratio.

6. The total velocity increases discontinuously across the leading edge and it might seem that the pressure should decrease correspondingly. There is, however, a sufficient increase of total energy to cause the pressure to increase. This is because the term ωrv is always greater than the term $\frac{1}{2}v^2$ in Equation (124).

7. The fact that the residuals are nearly constant on cylindrical surfaces extending downstream from the trailing edge of the blades means essentially that the tangential vorticity is nearly constant on these surfaces. This confirms, for this particular example at least, the linearizing assumption used by Marble (References 1 and

4), when he assumed in the axial flow problem that the vorticity was transported unchanged on cylindrical surfaces. It is surprising that this is also true in this region, bounded by cylindrical surfaces but located downstream of a complicated region of mixed flow.

IX. AN EXAMPLE OF SUBSONIC FLOW THROUGH AN ACTUATOR DISK

An example of the motion of an incompressible fluid acted upon by an infinite system of blades was presented in Part VIII. As regards rotationality effects, compressible fluid motion through a system of blades can be analyzed similarly, the main difference being the density ratio multiplying the vorticity term. Furthermore, it was shown in Part V-B that the flow in the region of the blades, where the vorticity is bound, is much less complicated than in the region downstream of the blades, when the vorticity is free. Therefore the present example, conceived in order to isolate the role compressibility plays in altering the rotationality effects, is concerned only with the more complicated upstream and downstream regions. The blade region is therefore concentrated into an "actuator disk" in which the tangential velocity jumps discontinuously from zero to some finite value. It is assumed that the "blade system" is stationary, hence the total energy of the fluid is constant throughout the field. The matching conditions for the meridional flow at the discontinuity are the same as stated in Part V-D for leading edge discontinuities.

The channel boundaries are concentric cylindrical surfaces, and at the entrance the axial velocity is constant and the radial and tangential velocities are zero. The jump in tangential velocity is

equal in magnitude to the inlet velocity of the incompressible example. For the downstream boundary condition the flow is independent of the axial coordinate. This problem is solved for two values of the inlet Mach number (0 and 0.2), the inlet mass-velocity and the tangential velocity being the same in both cases.

The applicable equations, obtained from Equations (63) and (65) with $\omega = 0$, $(rv)_1 = 0$, and E_1 = constant, are:

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = \psi_n (\ln \rho)_n \qquad \text{upstream}$$

$$= \psi_n (\ln \rho)_n - \frac{1}{2}\left(\frac{\rho}{\rho_*}\right)^2 \frac{d}{d\psi}\left(\frac{rv}{a_*}\right)^2 \qquad \text{downstream} \tag{125}$$

where $rv(\psi)$ and $\frac{d}{d\psi}\left(\frac{rv}{a_*}\right)^2$ are functions of ψ only and are evaluated at the downstream side of the discontinuity.

The boundary conditions are: ψ = constant on the cylindrical boundaries, $-\frac{1}{r}\psi_r = 1$ on the upstream boundary, and $\psi_z = 0$ on the downstream boundary. For each approximation to $rv(\psi)$ the downstream boundary condition can be modified so that ψ is prescribed, as in the previous example (Part VIII-B).

A) Incompressible Flow Through the Actuator

The difference equation corresponding to Equation 125 for incompressible flow is:

$$\psi_1 + \left(1 - \frac{\delta}{2r}\right)\psi_2 + \psi_3 + \left(1 + \frac{\delta}{2r}\right)\psi_4 - 4\psi_0 = 0 \qquad \text{upstream}$$

$$= -\frac{1}{2}\frac{d}{d\psi}(rv)^2 \delta^2 \qquad \text{downstream} \tag{126}$$

The inlet velocity is taken as 1000 and the boundary conditions are:

$\psi = 500\,(9 - r^2)$ on upstream boundary

$\psi = 0$ on $r = 3$

$\psi = 4000$ on $r = 1$

$\frac{\partial \psi}{\partial z} = 0$ on downstream boundary

The influence coefficients and the net points of the flow field are shown to scale in Figure 22.

The first approximation to $rv(\psi)$ was based on irrotational flow within the same boundary. That is, it was assumed that the upstream value of ψ held throughout the field. The rv so obtained is shown in Figure 23. Before relaxation, the corresponding downstream boundary values were further improved by assuming half the change in ψ occurred upstream of the actuator, thus obtaining an approximate value of ψ at the actuator and consequently a new $rv(\psi)$. The boundary values corresponding to this last rv were then used for the first complete relaxation, after which $rv(\psi)$ was found to be unchanged. The result of this complete relaxation was therefore the final solution of the problem.

Values of the axial, radial, and tangential velocities at several representative stations are shown in Figures 24, 25, and 26. Lines of constant static pressure are shown in Figure 27. Far downstream of the actuator the axial velocity is much greater near the hub than near the shroud. The maximum value of the radial velocity occurs just below the "center' of the channel. The pressure is

discontinuous across the actuator, and most of the pressure changes occur downstream of the actuator.

Perhaps the most significant feature of the solution is that the radial velocity and the changes in axial velocity are approximately symmetrical about the actuator. It is clear, however, that they are not exactly symmetrical since the forcing function is not symmetrical. One consequence of linearization (Reference 4) is that the meridional flow is symmetrical about the discontinuity.

The solution of this example will be discussed and compared with the compressible solution of the next section.

B) Compressible Flow Through the Actuator-$M_{inlet} = 0.2$

The difference equation corresponding to Equation (125) is:

$$\psi_1 + \left(1-\frac{\delta}{2r}\right)\psi_2 + \psi_3 + \left(1+\frac{\delta}{2r}\right)\psi_4 - 4\psi_o$$

$$= \left[\psi_n (\ln\rho)_n\right]\delta^2 \qquad \text{upstream} \qquad (127)^*$$

$$= \left[\psi_n (\ln\rho)_n - \frac{1}{2}\left(\frac{\rho}{\rho_t}\right)^2 \frac{d(rv)^2}{d\psi}\right]\delta^2 \qquad \text{downstream}$$

The boundary conditions for the streamfunction* were essentially the same as for the previous example (IX-A). On the upstream boundary $-\frac{1}{r}\psi_r = \frac{\rho}{\rho_o} w = 1000$, and $M = 0.2$. Other parameters on the upstream boundary were:

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} = 1.020$$

$$\frac{a}{a_o} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}} = 1.004$$

$$w = 1000\,\frac{\rho_o}{\rho} = 1020$$

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* For this example only, the streamfunction is defined by:

$$u = \frac{\rho_o}{\rho}\frac{1}{r}\psi_z \;\; ; \;\; w = -\frac{\rho_o}{\rho}\frac{1}{r}\psi_r$$

$$a = \frac{W}{M} \qquad = 5100$$

$$a_0 \qquad = 5080$$

The tangential velocity jumped discontinuously to 1000 in the actuator.

The influence coefficients and the net points of the field are the same as in the previous example (Figure 22).

The first step in obtaining a solution was to estimate as accurately as possible, by any means whatever, the right side of the equation. This was done as follows: The vorticity term, $\frac{1}{2}\frac{d}{d\psi}(rv)^2$ was very closely approximated by simply using the value from the incompressible solution just obtained. The extreme accuracy of this choice is indicated in Figure 28, where the initial and final values of rv(ψ) are plotted. The first estimation of the density term was likewise based on the irrotational velocities, but only values on the downstream boundary were calculated first. When this was done it was clear that the streamlines deflected less when compressibility was present, for the compressibility term, $\psi_n(\ln\rho)_n\delta^2$, was negative and subtracted from the vorticity term, $\left[-\frac{1}{2}\left(\frac{\rho}{\rho_0}\right)^2\frac{d}{d\psi}(rv)^2\right]$ which was positive. In addition $\left(\frac{\rho}{\rho_0}\right)^2$, which decreased according to the increase in total velocity downstream of the actuator, also acted to reduce the right side of the equation. However, the right side of the equation was estimated on the downstream boundary using the density term just calculated, and the streamfunction and the corresponding density were calculated there. These values were treated

as downstream boundary values for the first complete relaxation. In order to complete the estimation of the residuals throughout the field the density changes calculated from the irrotational streamlines were corrected by a factor chosen to give the correct density on the downstream boundary. In this way it was possible to make a rather accurate estimate of the right side of the equation, and the desired residuals for the first complete relaxation. From the results of the first relaxation the densities and velocities, and finally new desired residuals, were obtained. The new desired residuals agreed very well with the previous ones except in regions where the Mach number was greatest. Small changes of the stream-function brought the residuals into satisfactory agreement and led to the final solution, presented in Figures 28 through 32.

The velocities are expressed in terms of the inlet axial velocity, rather than a velocity of sound, for comparison with the incompressible solution. The Mach number based on the total velocity is shown in Figure 32. The velocity components are similar to those of the incompressible solution, except of course for the discontinuities at the actuator.

C) A Discussion of the Solutions

In the two foregoing examples an otherwise axial flow is distorted by vorticity generated by a so-called actuator disk. The strength of the vorticity and the boundary conditions of the flow are

the same in both examples, yet the solutions, although generally similar, differ distinctly in at least two ways: the total deflection of the flow brought about by the vorticity is decidedly less when the flow is compressible; the axial variations of the compressible flow are concentrated more in the vicinity of the actuator, the concentration being more intense where the Mach number is higher.

The first point of difference was discussed in the previous section and is confirmed by Figure 33, where the streamlines of the two solutions are compared. A general conclusion set forth in Part VI is that the local effect of vorticity, as regards deflection of the streamsurfaces, reduces to zero as the "governing" Mach number increases to one, and then increases, but with opposite sense, as the Mach number increases above one. The second point of difference could probably be predicted by the Prandtl-Glauert similarity transformation, applied to linearized equations. Its occurrence here is evident in Figure 32, and to some extent in Figures 25 and 30 where the maximum of the radial velocity for the compressible flow occurs nearer the center of the channel, indicating more rapid axial changes near the hub where the Mach number is larger.

X. EXAMPLES OF TRANSONIC FLOW THROUGH AN ACTUATOR DISK

This section provides examples of flow through actuator disks with maximum Mach numbers near one. Two examples, with the same inlet Mach number (0.555), but with actuator disks of different strength, are considered. In one example the maximum Mach number attained is one, in the other, 1.12. The actuator model is the same as described in Part IX but the tangential velocity induced at the actuator is not constant there. Instead the tangential velocity is prescribed on the downstream boundary, and the jump at the actuator is not known until the problem is solved. With this prescription it is possible to replace the downstream boundary condition, $\frac{\partial \psi}{\partial z} = 0$, with the condition that the streamfunction itself is prescribed, for the downstream conditions can be determined without knowing the intermediate flow. The limiting flow far downstream of the blades was discussed in Part V-F. For the particular case where the tangential velocity is constant, Figures 43 and 44 show the maximum Mach number as a function of the inlet Mach number and the tangential velocity. It was also shown in Part V-F that there is a maximum tangential velocity that can be imposed by an actuator disk and that this maximum corresponds to the choking condition. The limiting conditions for these examples are indicated by the two points shown in Figure 43.

The differential equation (Equation 125)) of the previous examples is applicable here but the definition of the streamfunction* is slightly different. The difference equation for both examples is (from Equation (111)):

$$\psi_1 + \left(1-\frac{\delta}{2r}\right)\psi_2 + \psi_3 + \left(1+\frac{\delta}{2r}\right)\psi_4 - 4\psi_0$$

$$= \left[\psi_n (\ln \rho)_n\right]\delta^2 \qquad \text{upstream} \qquad (128)$$

$$= \left[\psi_n (\ln \rho)_n - \frac{1}{2}\left(\frac{\rho}{\rho_*}\right)^2 \frac{d}{d\psi}\left(\frac{rv}{a_*}\right)^2\right]\delta^2 \qquad \text{downstream}$$

A) Flow with Maximum Mach Number of One

The following conditions are prescribed:

Inlet Mach number: M = .555 , M* = .584

Tangential velocity: v = 0 at inlet

$\left(\frac{v}{a_*}\right)^2 = .21$ on downstream boundary

and calculated from these are:

Maximum axial velocity: $\frac{w}{a_*} = .888$

Maximum Mach number: $\left(\frac{w^2+v^2}{a^2}\right)^{\frac{1}{2}} = 1$

The influence coefficients for each point in the net and the boundary values for ψ are shown in Figure 22.

The method of solution was essentially the same as described in the previous example. In determining the approximate residuals for the first relaxation the value of the rotationality term $\frac{1}{2}\frac{d}{d\psi}\left(\frac{rv}{a_*}\right)^2$ obtained on the downstream boundary was assumed constant

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* For these examples the streamfunction ψ is defined as:

$$\frac{u}{a_*} = \frac{\rho_*}{\rho}\frac{1}{r}\psi_z \quad ; \quad \frac{w}{a_*} = -\frac{\rho_*}{\rho}\frac{1}{r}\psi_r$$

on cylindrical surfaces extending upstream to the actuator. The density throughout the field was estimated using the known values of the density upstream and downstream and the approximate magnitude of the density discontinuity at the actuator. The first desired residuals were obtained in this way and a complete relaxation performed. The final solution then followed simple iteration in which the desired residuals were obtained from the solution of the previous step. After two complete relaxations the residuals were in good agreement except in the region where the Mach number was largest. Minor modification then led to the final solution.

The axial and radial velocities are shown for representative stations in Figures 35 and 36. The Mach number based on the total velocity is shown in Figure 37.

This solution exhibits the properties already discussed in the previous compressible example of Part IX, as noted in the comparison of the next section.

B) Flow with Maximum Mach Number of 1.12

The following conditions are prescribed:

Inlet Mach number: $M = .555$, $M^* = .584$

Tangential velocity $v = 0$ at inlet

$\left(\frac{v}{a_*}\right)^2 = .264$ on downstream boundary

and calculated from these are:

Maximum axial velocity = $\frac{w}{a_*}$ = .966

Maximum axial Mach number = $\frac{w}{a}$ = .935

Maximum total Mach number = $\left(\frac{w^2+v^2}{a^2}\right)^{\frac{1}{2}}$ = 1.12

The influence coefficients for each point of the net and the boundary values for ψ are shown in Figure 38. In addition the velocity, density, etc., at the downstream boundary, are recorded in Table II.

The residuals were first estimated using the meridional streamlines of the previous solution. A complete relaxation was performed and the iteration started. Unfortunately the densities obtained from the second complete relaxation were imaginary and straightforward iteration could not be continued. To remedy this the axial mass-velocity, $\frac{1}{r}\psi_r$, was changed just enough to make the densities real. Then the iteration could be continued. It was observed that the numerical solution of this example was less accurate than the solution of the previous example, although the net points (Figures 34 and 38) were four times as dense. This is due to the nearly vertical tangent of the density curve (Figure 42) when the meridional velocity is nearly sonic, and to the fact that the radial derivative of the streamfunction could not be accurately determined on the inner radius where the large Mach numbers occur.

The axial and radial velocities are shown in Figures 39 and 40, the Mach numbers in Figure 41. The solution exhibits the same properties as have already been described in the preceding examples.

It is surprising that the streamsurfaces are deflected less in this second example where the tangential velocity is greater. This can be seen by comparing the downstream values of the streamfunction in Figures 34 and 38, and indicates that there is a certain tangential deflection for which the deflection of the streamsurfaces is maximum.

XI. CONCLUSION

A theory has been developed which permits the analysis of compressible flow in turbomachines having infinitely many blades when the governing velocity is subsonic. Several examples, solved by the method of finite differences, have been presented. The fundamental idea underlying the theory is that the force field representing the infinitely many blades is necessarily a "pseudo-conservative" field. Because of this the three components of the field can be expressed in terms of two functions, one describing the input of energy, the other the shape of the blades. The functions which must be prescribed and the boundary conditions which must be imposed are then quite clear, and the heretofore more difficult direct problem becomes relatively easy.

Two principal effects of compressibility were noted: the deflection of the streamsurfaces brought about by a given vorticity distribution decreased as the Mach numbers (subsonic) were increased; the streamwise variations of the flow became more concentrated as the Mach numbers were increased.

The foregoing theory is limited in that the fluid must be non-viscous and the number of blades must be infinite. In addition the application of the theory when the "governing" velocity is supersonic is questionable. It is clear that non-viscous flow under the action of a finite number of blades must be completely understood

before the viscous problem is attacked. Viscosity has no place in problems in which the number of blades is infinite, for the blades do not act as boundaries of the flow.

The extension to machines with a finite number of blades could conceivably be carried out by three-dimensional finite difference methods but the analysis would be very lengthy. A first order approximation to the pressure and total energy is indicated in the appendix. Perhaps the least understood problems are those in which the governing velocity is transonic. The equations were developed and discussed generally in Part VI. A suggested approach might be to consider only the region in and upstream of the blades, where the vorticity effects are linear. Prescription of radial blades would allow further simplification by means of the transformation given in Equation (121). Then a perturbation equation analogous to the transonic equations could be developed. Any analysis which will lead to a better understanding of the cushioning effect mentioned in Part VI would be a worthy contribution.

APPENDIX

A First Order Approximation to the Flow with a Finite Number of Blades

A first order approximation to the flow with a finite number of blades can be obtained from the solution with an infinite number of blades by replacing the body force field acting in a sector between two blades with a pressure gradient force, thus essentially reversing the reasoning which first led to the assumption of infinite blades. Then integration of the pressure gradient from one blade to another will give the circumferential pressure distribution between the blades.

The mean value of the pressure, as obtained from the analysis with an infinite number of blades, is denoted by p_m. Let p' denote the deviation of the total pressure from the mean: $p = p_m + p'$. The force field is then replaced by the pressure gradients as follows:

$$F_r = -\lambda f_r \quad = -\frac{1}{\rho}\frac{\partial p'}{\partial r}$$

$$F_\theta = \lambda \frac{1}{r} \quad = -\frac{1}{\rho}\frac{\partial p'}{\partial \theta}$$

$$F_z = -\lambda f_z \quad = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

If there are n blades distributed uniformly around the axis, the second equation can be integrated from one blade to the next to give the pressure discontinuity across a blade:

$$\frac{\delta p}{\rho} = -\lambda(\delta\theta) = \lambda\frac{2\pi}{n}$$

Because λ is independent of θ, the tangential variation of the pressure is linear between blades. The deviation from the mean pressure can therefore be easily determined at all points in the region of the

blades. The corresponding approximation of the total energy, ω rv, can be obtained from a form of Equation (31), using the velocities of the solution with an infinite number of blades.

It is significant that the equilibrium of the fluid has not been disturbed by the introduction of the pressure force for the force field, nor has fulfillment of the continuity requirement been altered. Actually the only assumption involved is that the tangential variation of the pressure is linear, i.e., λ is independent of θ. This first order approximation therefore seems very reasonable, particularly when there are many blades, and may be satisfactory in other cases. Unfortunately exact solutions with a finite number of blades are not available for determining the accuracy of the approximation.

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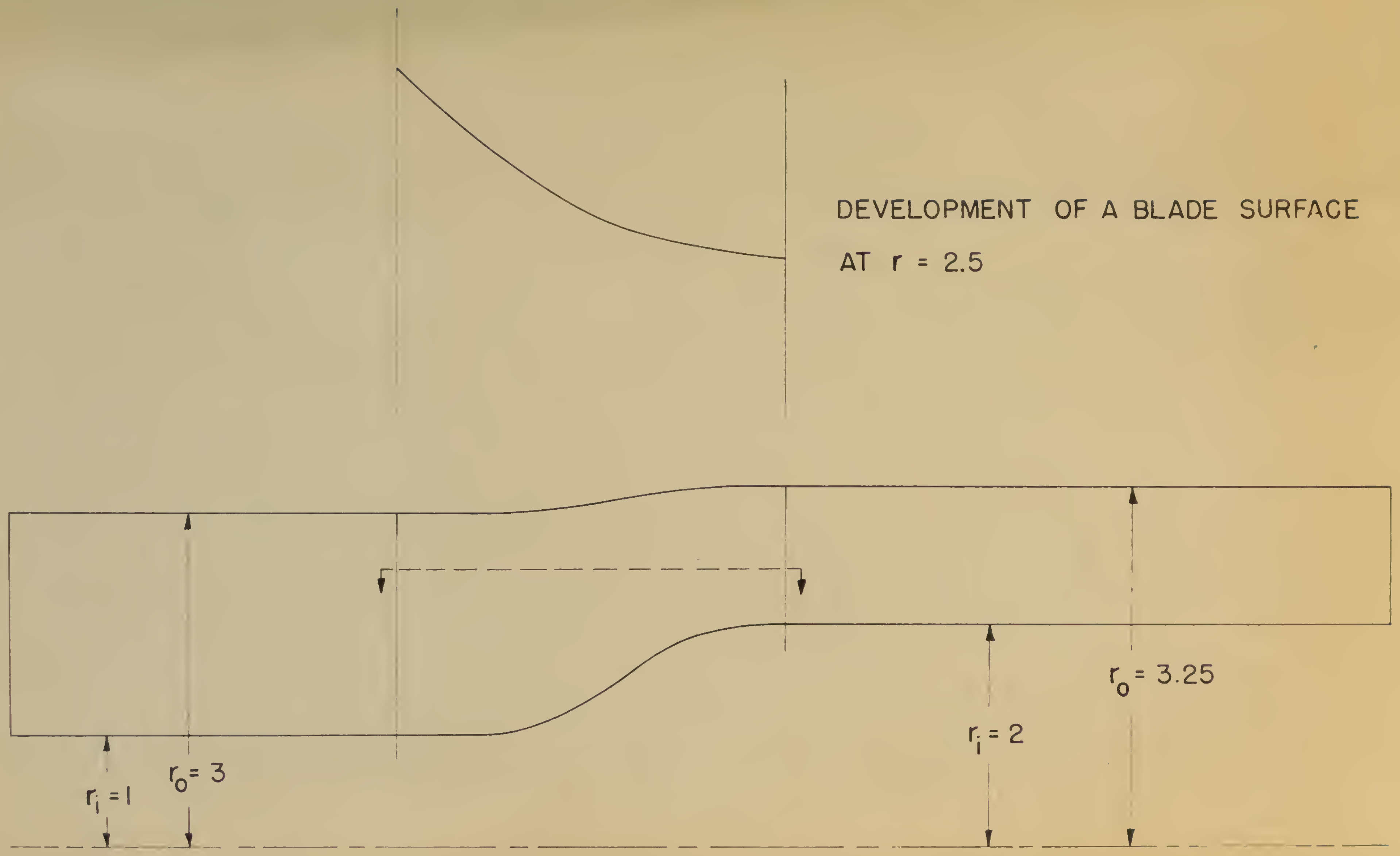


FIG. 8

HUB AND SHROUD CONTOURS - BLADE SHAPE

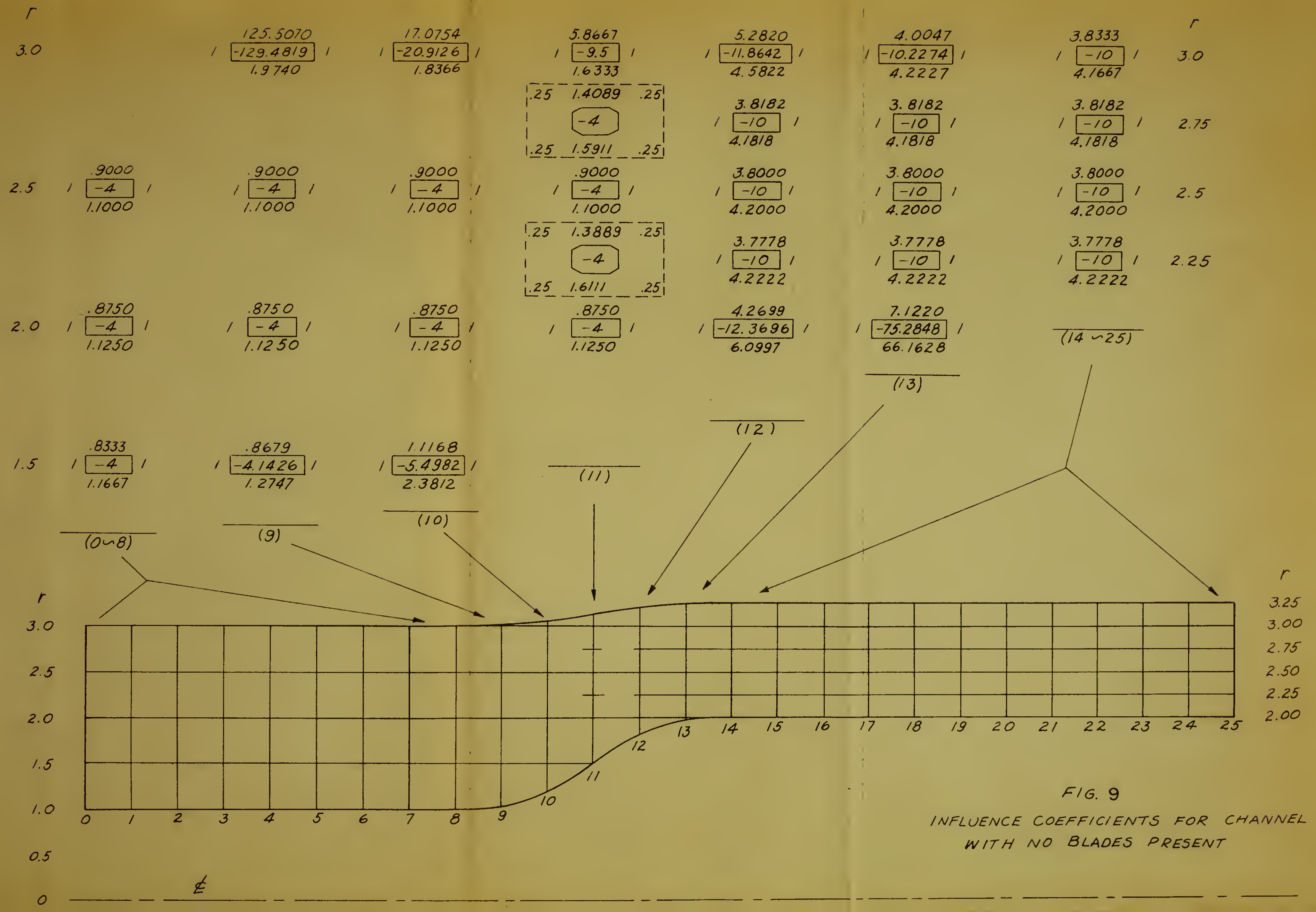
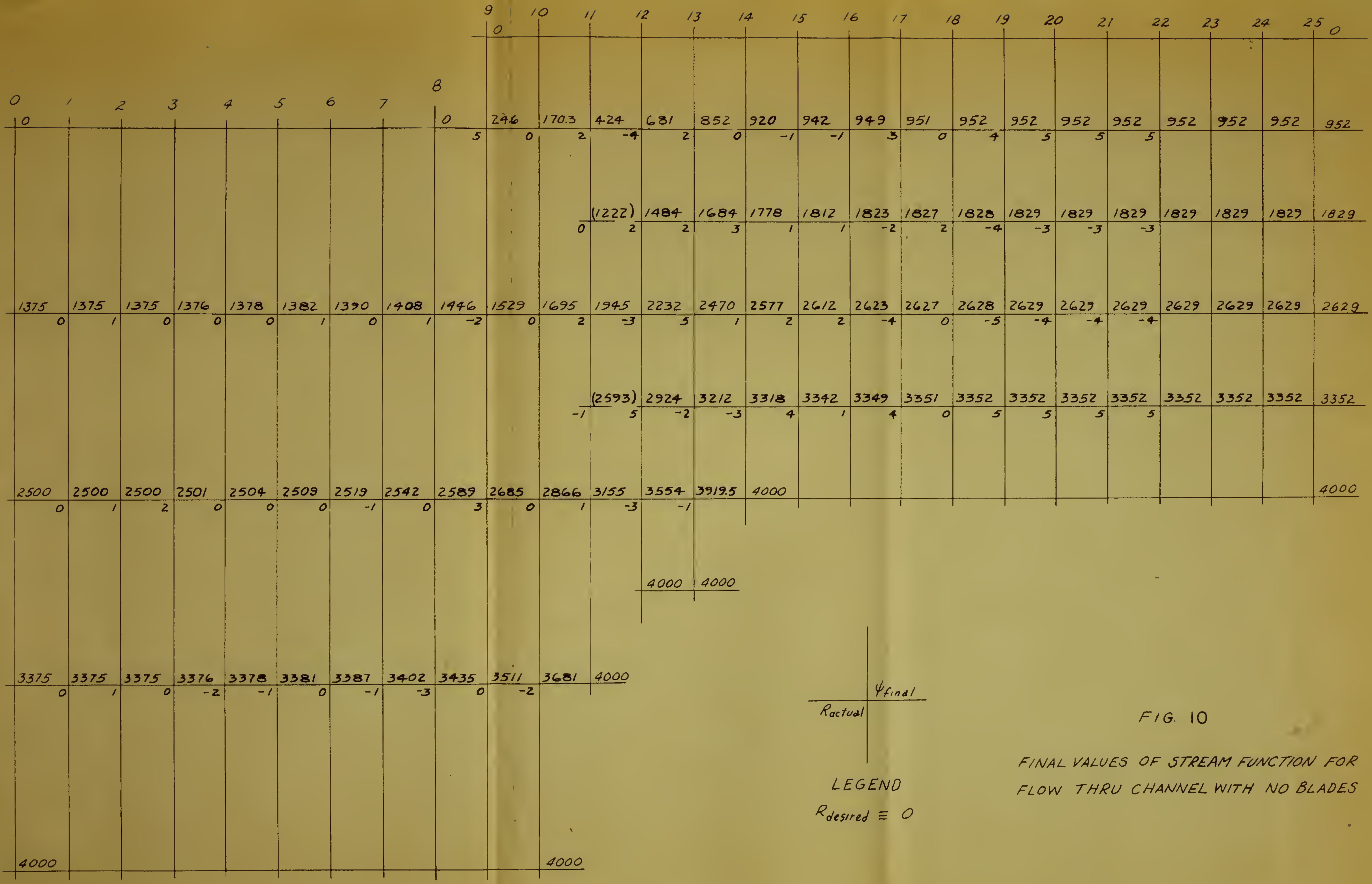


FIG. 9

INFLUENCE COEFFICIENTS FOR CHANNEL WITH NO BLADES PRESENT



LEGEND

$R_{desired} \equiv 0$

FIG. 10

FINAL VALUES OF STREAM FUNCTION FOR FLOW THRU CHANNEL WITH NO BLADES

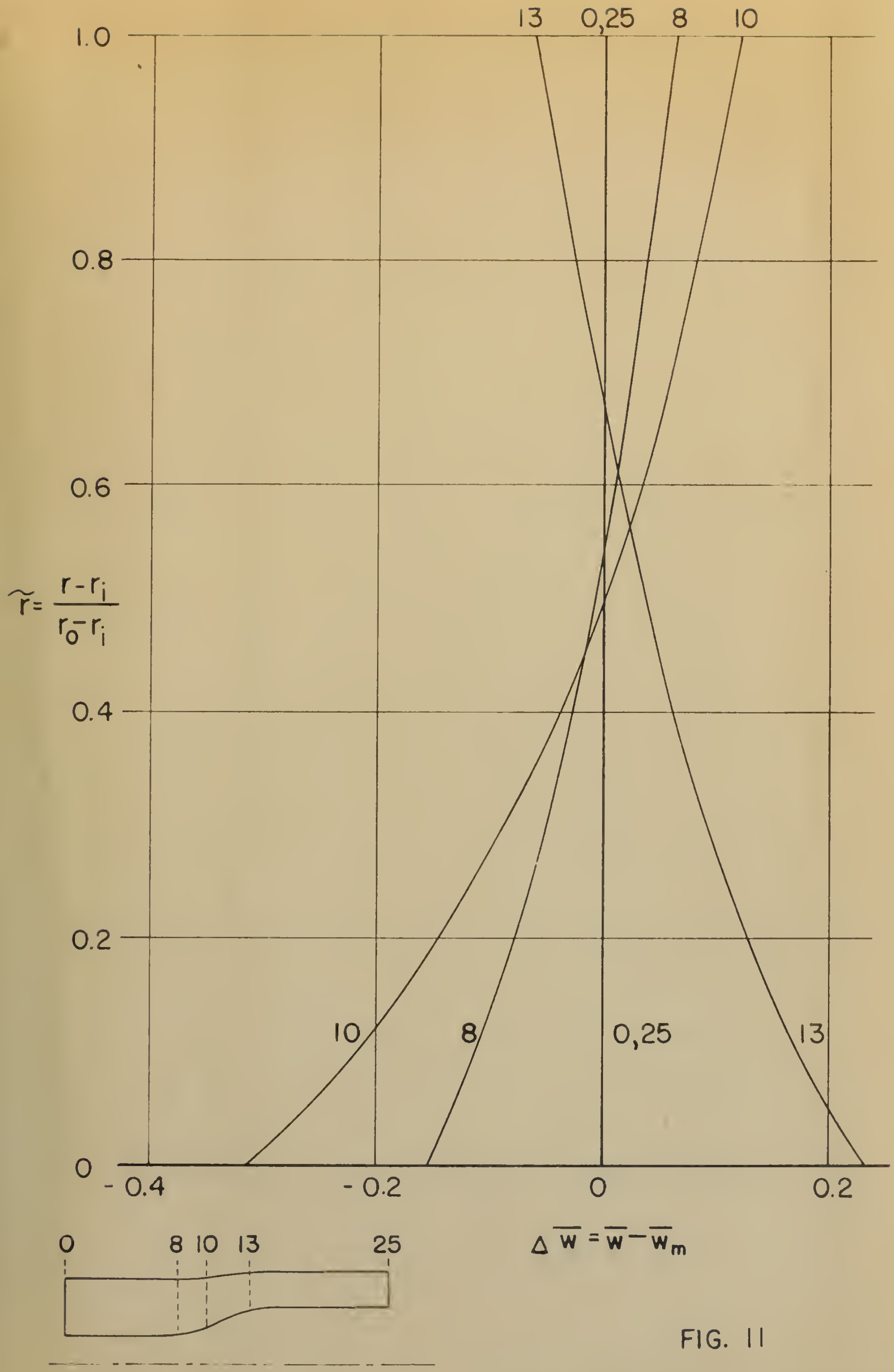
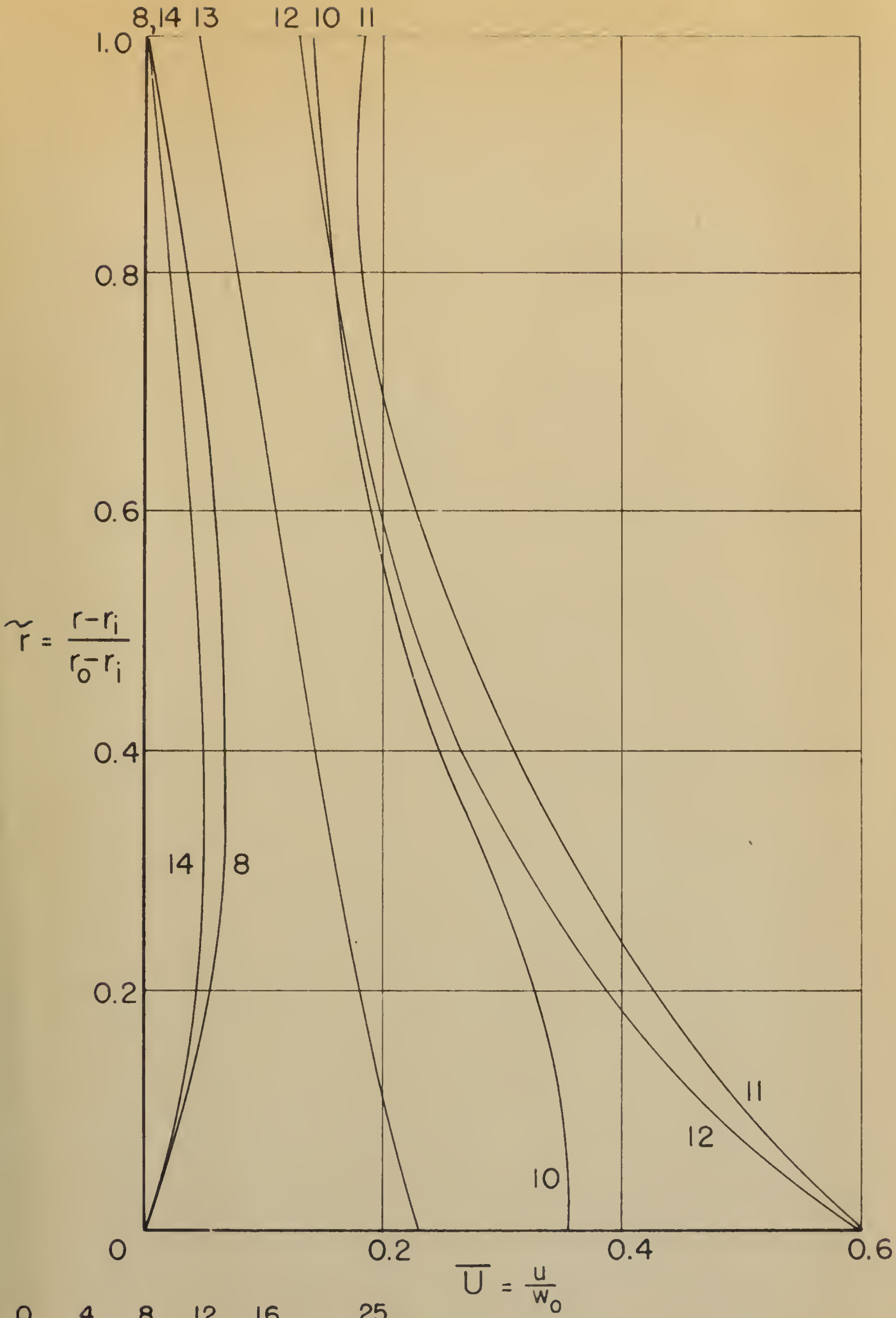


FIG. 11

AXIAL VELOCITY – CLEAR CHANNEL



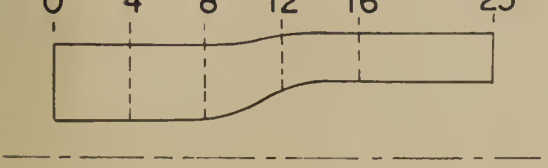


FIG. 12

RADIAL VELOCITY-CLEAR CHANNEL

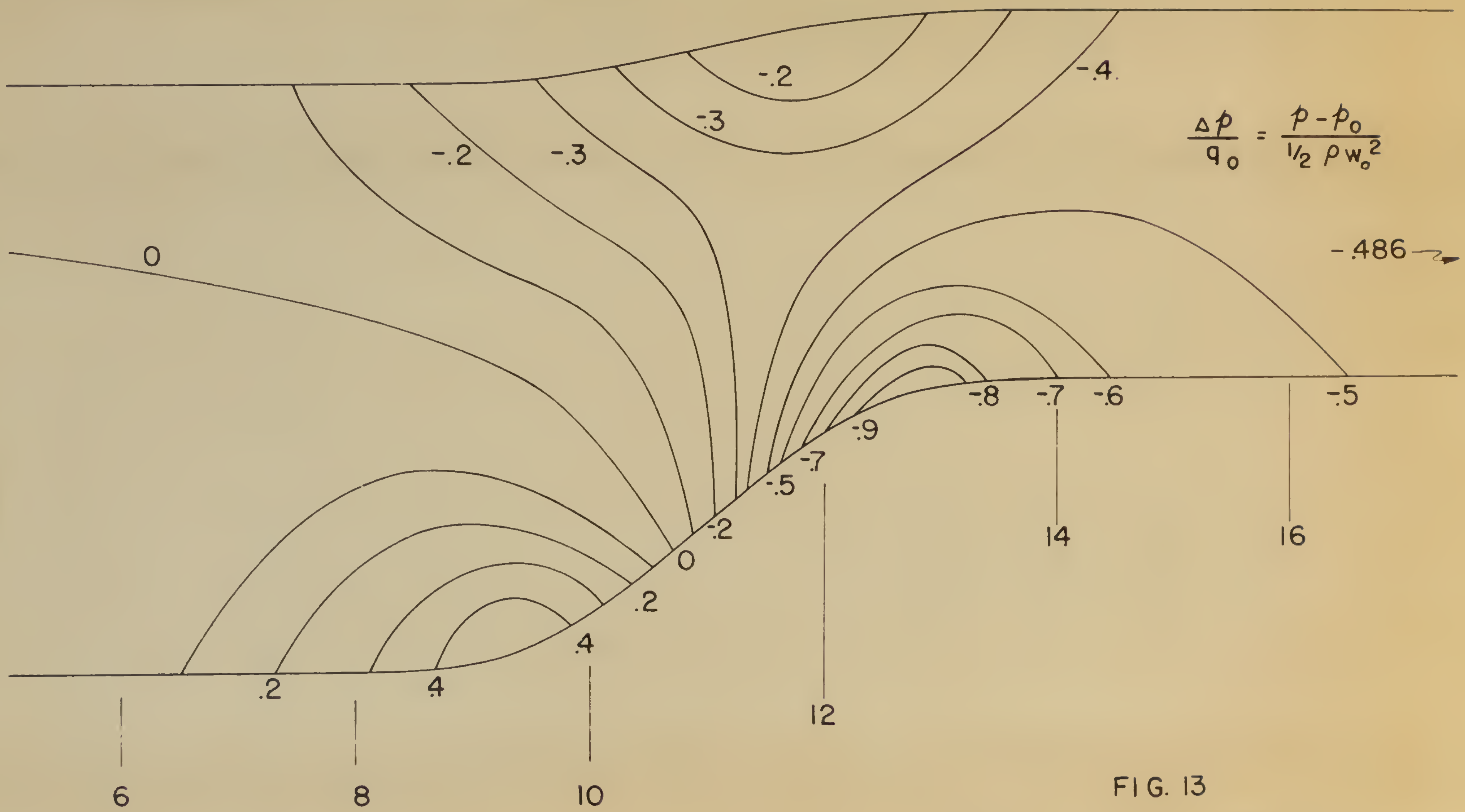


FIG. 13

LINES OF EQUAL PRESSURE
CLEAR CHANNEL

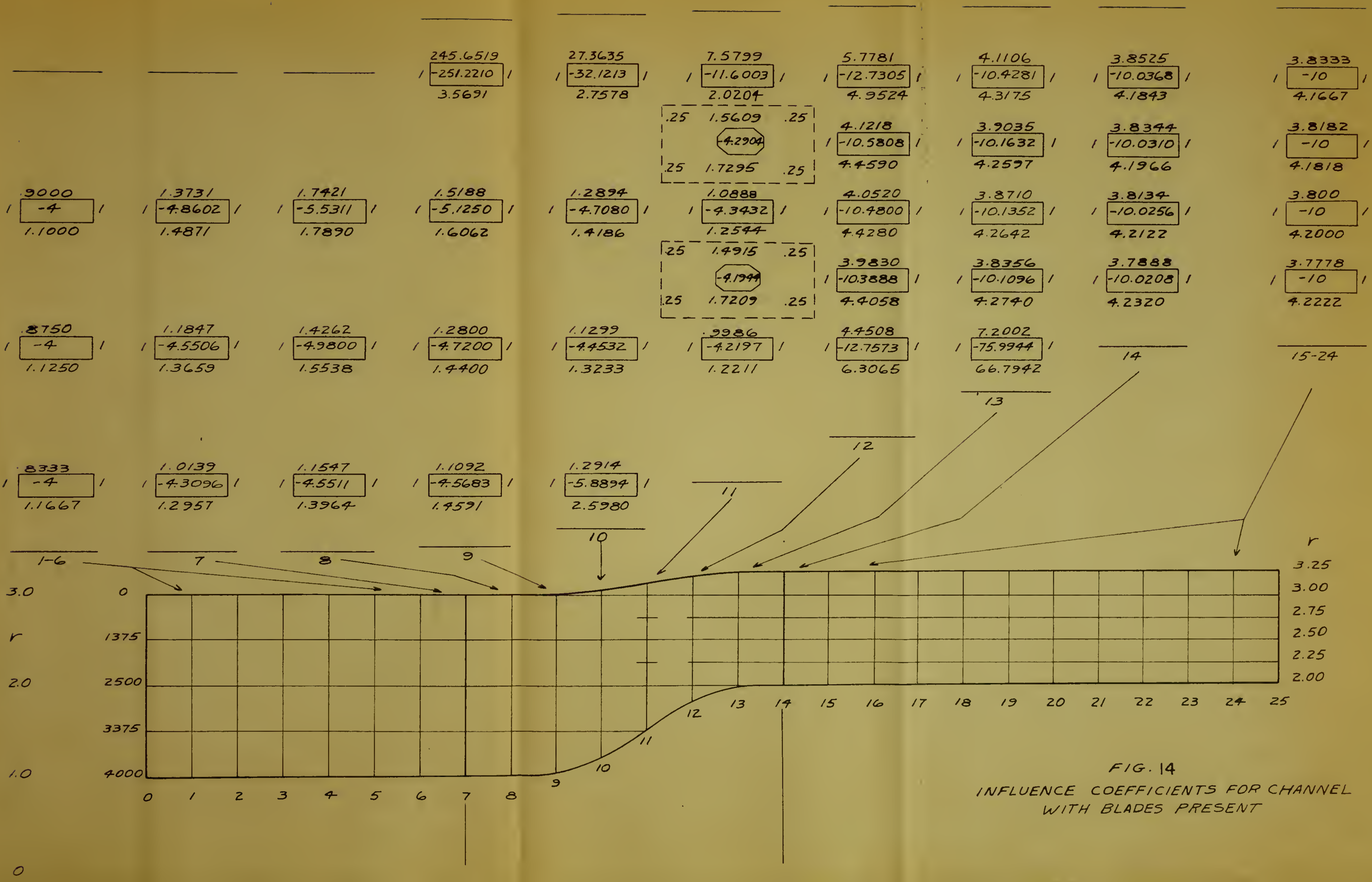


FIG. 14

INFLUENCE COEFFICIENTS FOR CHANNEL WITH BLADES PRESENT

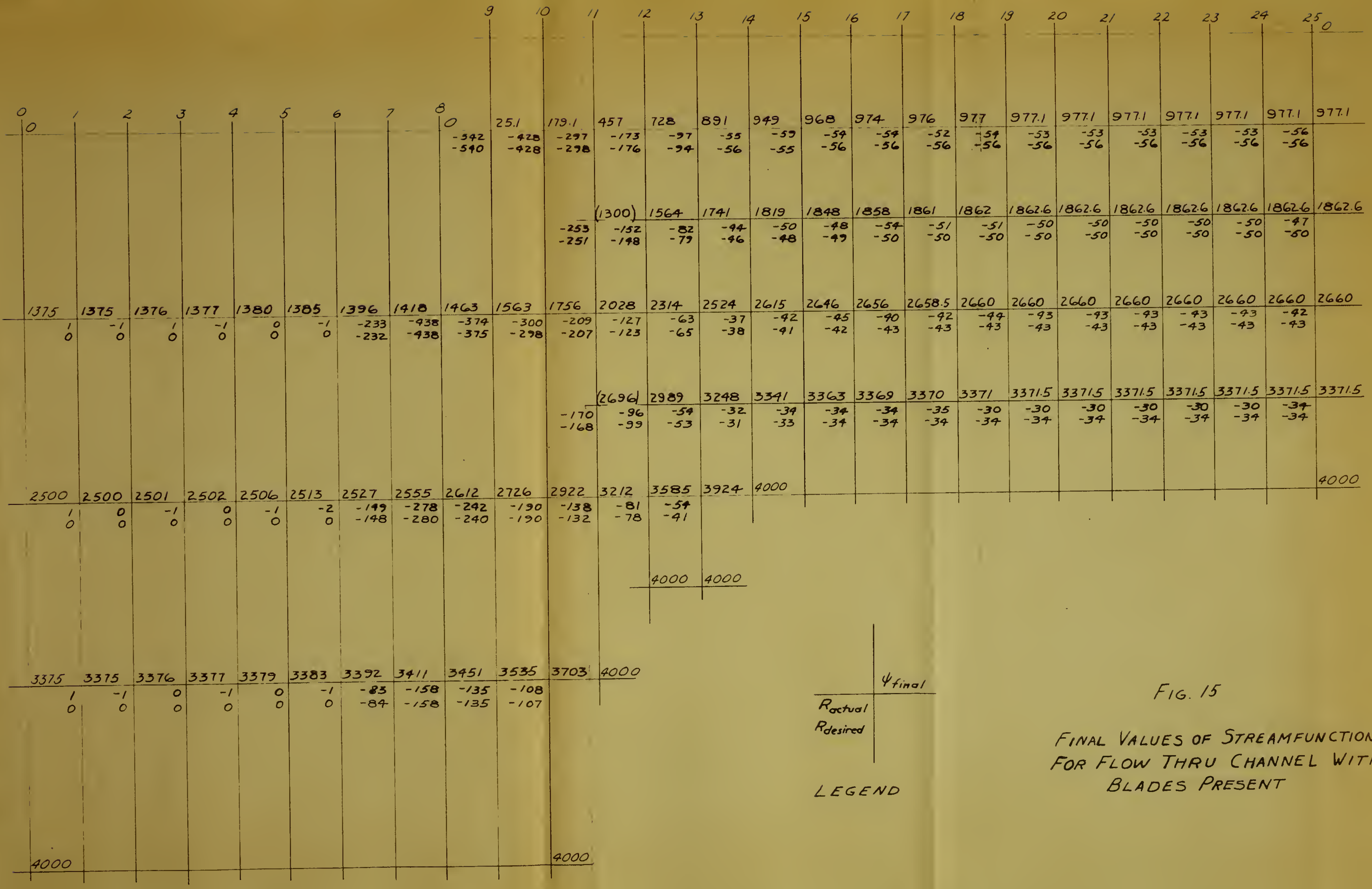


FIG. 15

FINAL VALUES OF STREAMFUNCTION FOR FLOW THRU CHANNEL WITH BLADES PRESENT

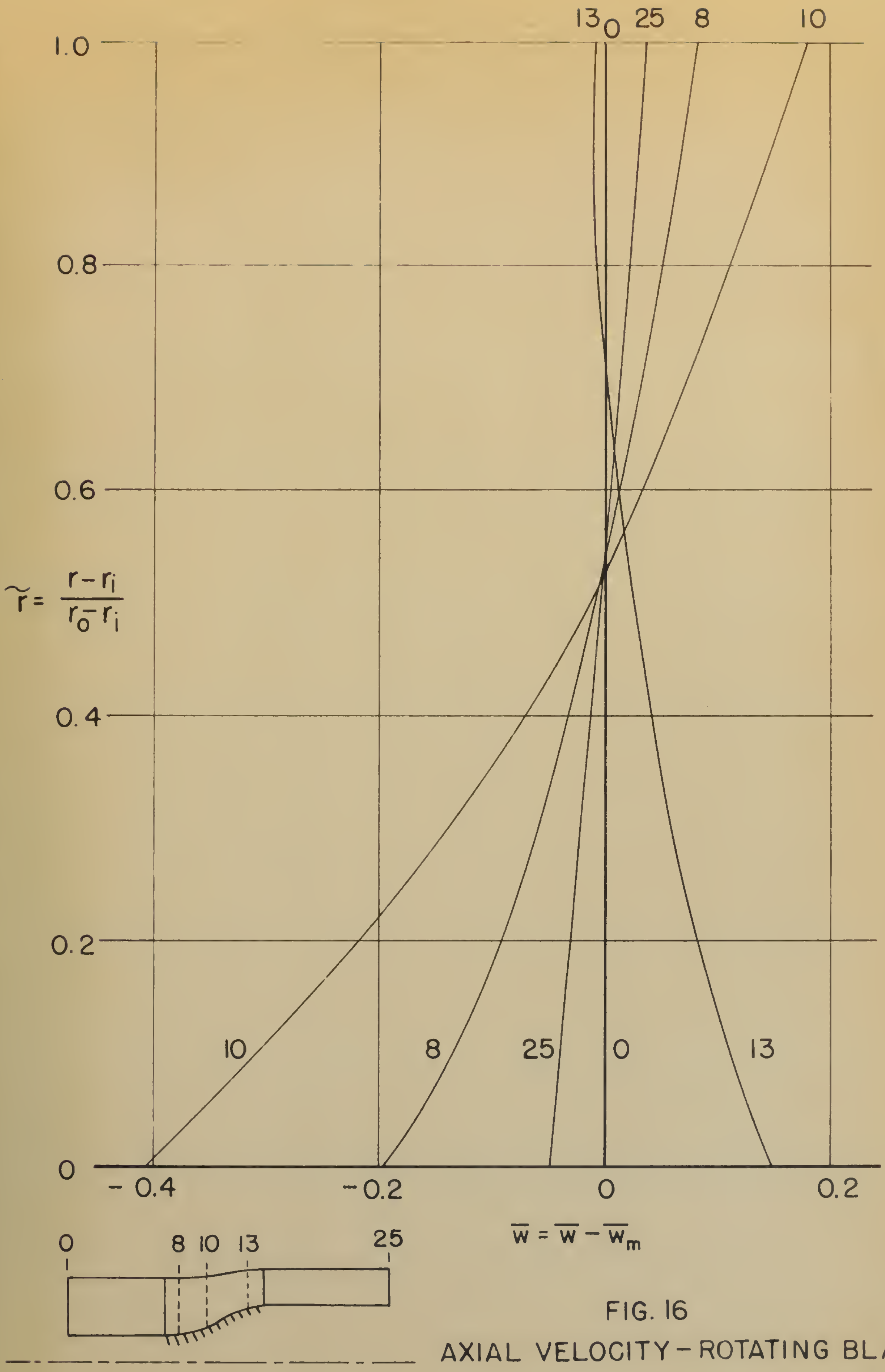


FIG. 16

AXIAL VELOCITY - ROTATING BLADES

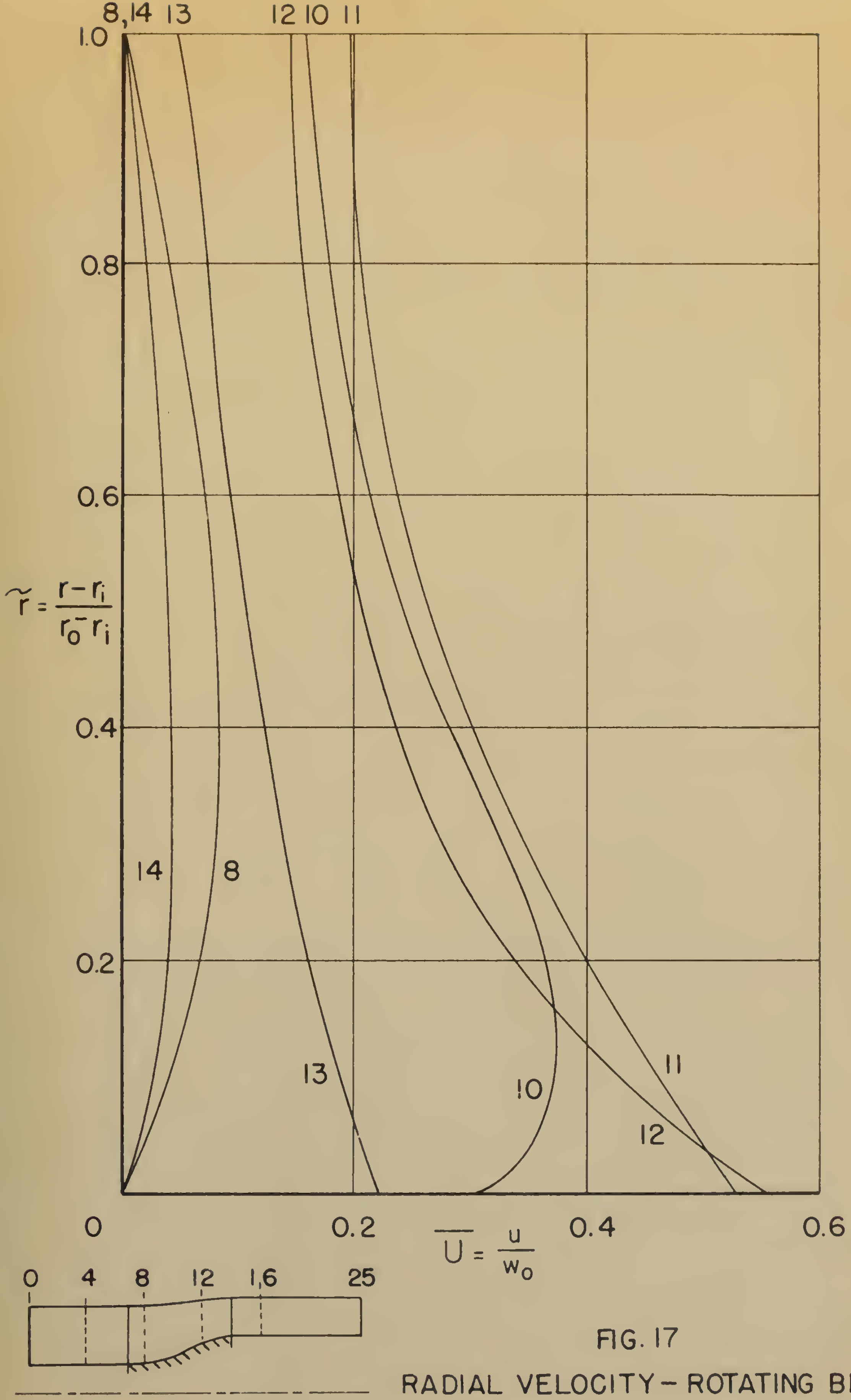


FIG. 17

RADIAL VELOCITY – ROTATING BLADES

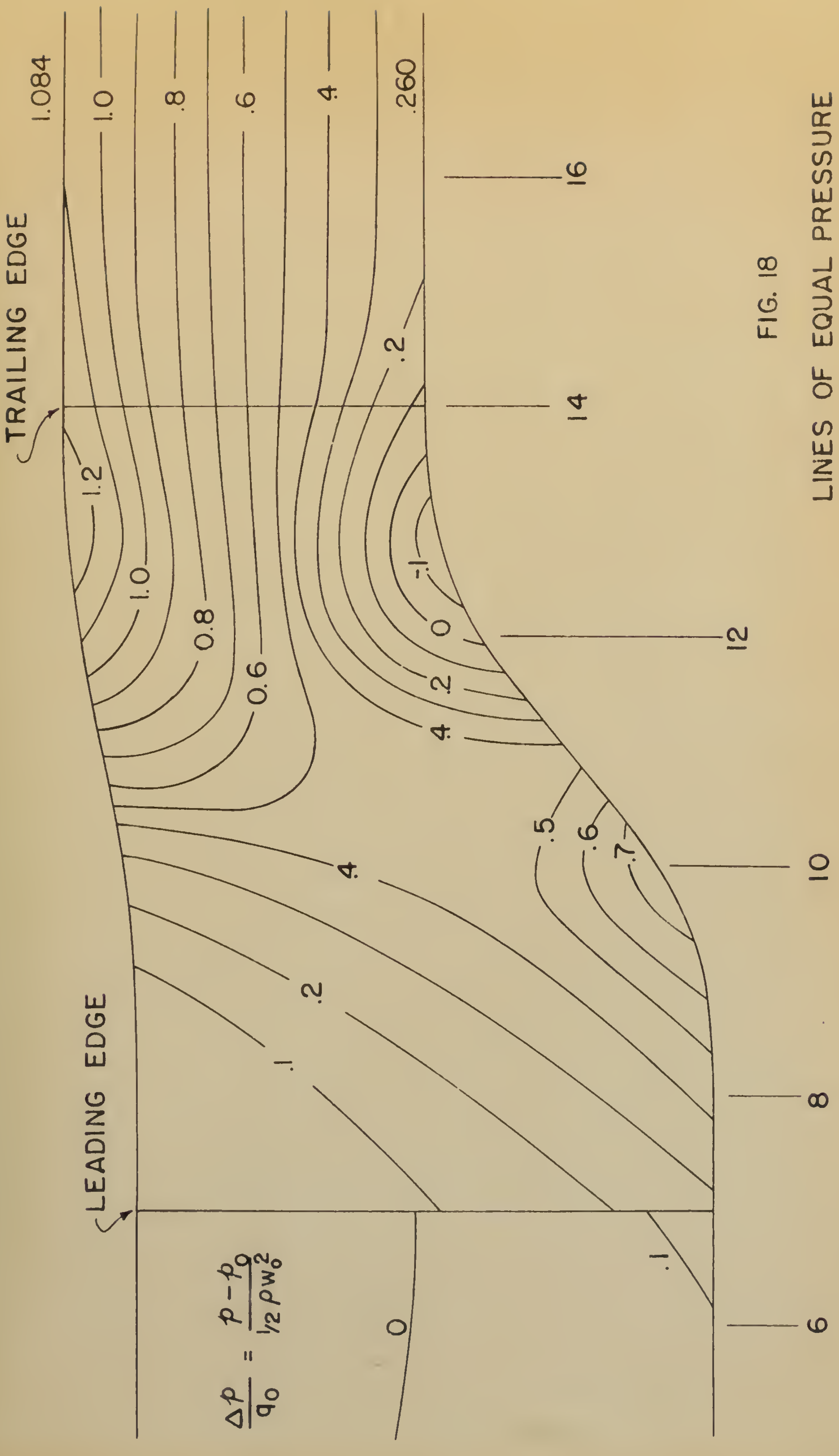


FIG. 18

LINES OF EQUAL PRESSURE

ROTATING BLADES

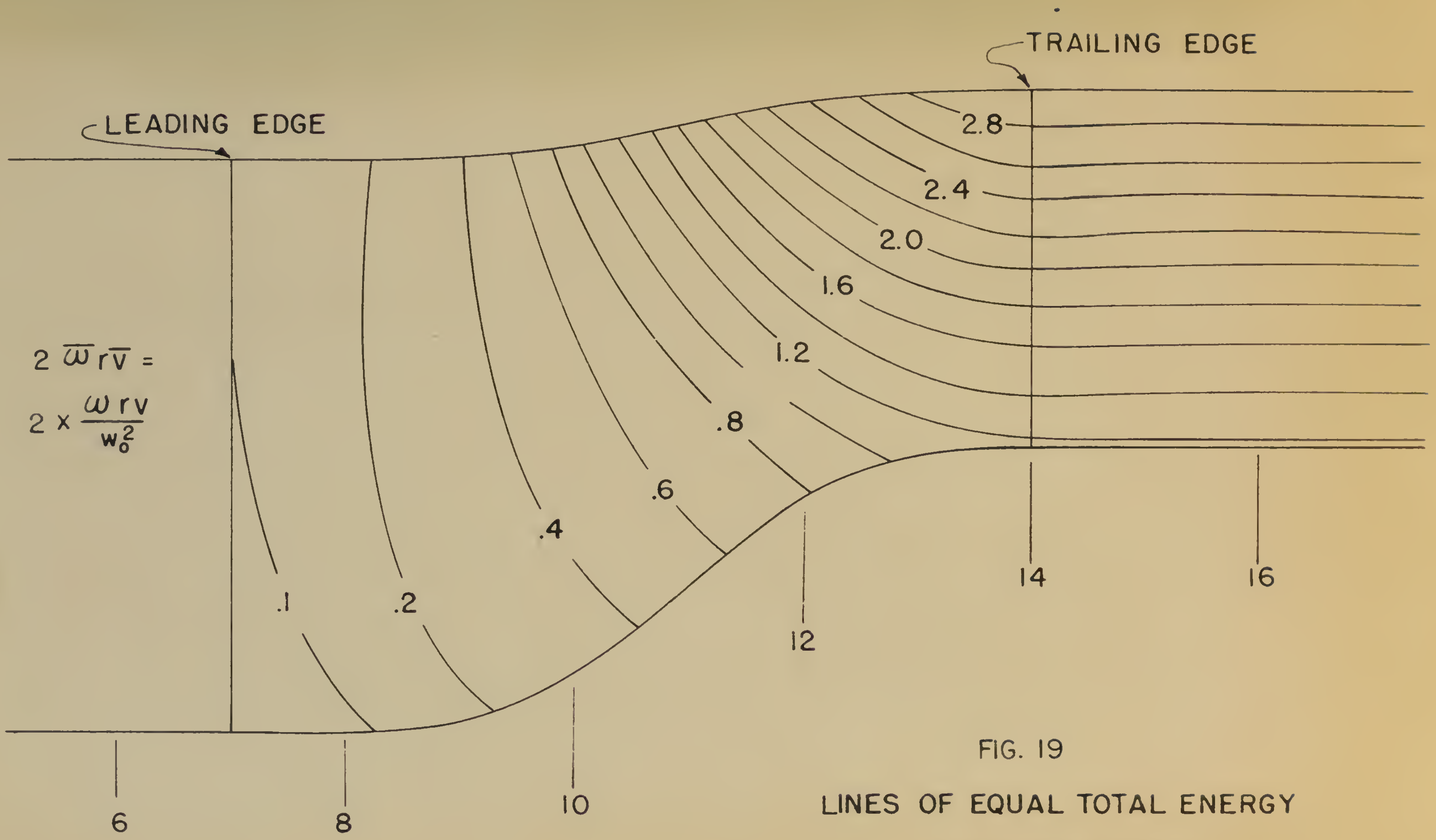


FIG. 19

LINES OF EQUAL TOTAL ENERGY

ROTATING BLADES

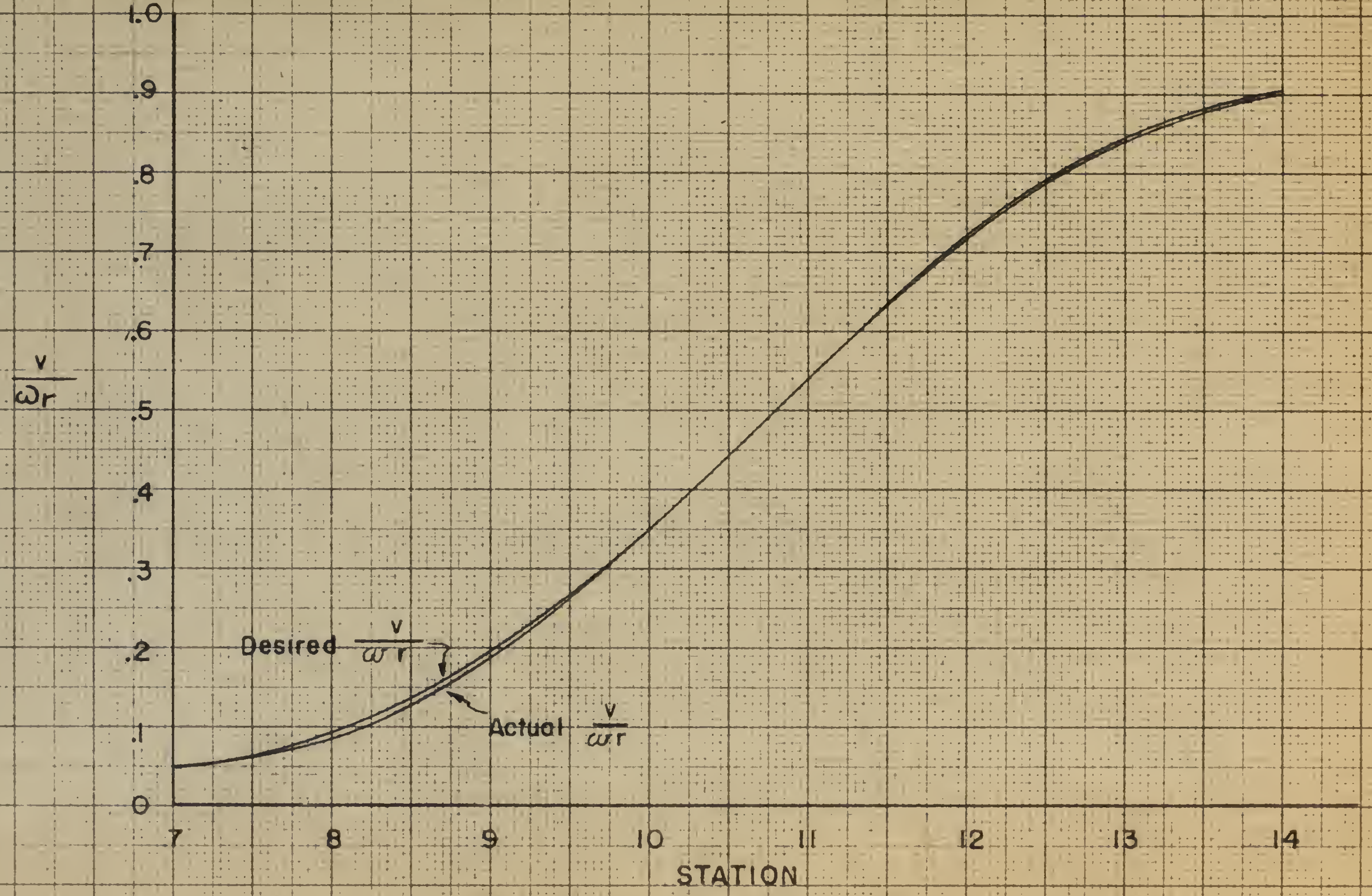


FIG. 20

DESIRED AND ACTUAL TANGENTIAL VELOCITY ON RADIUS $r = 2.5$

INCOMPRESSIBLE MIXED-FLOW PROBLEM

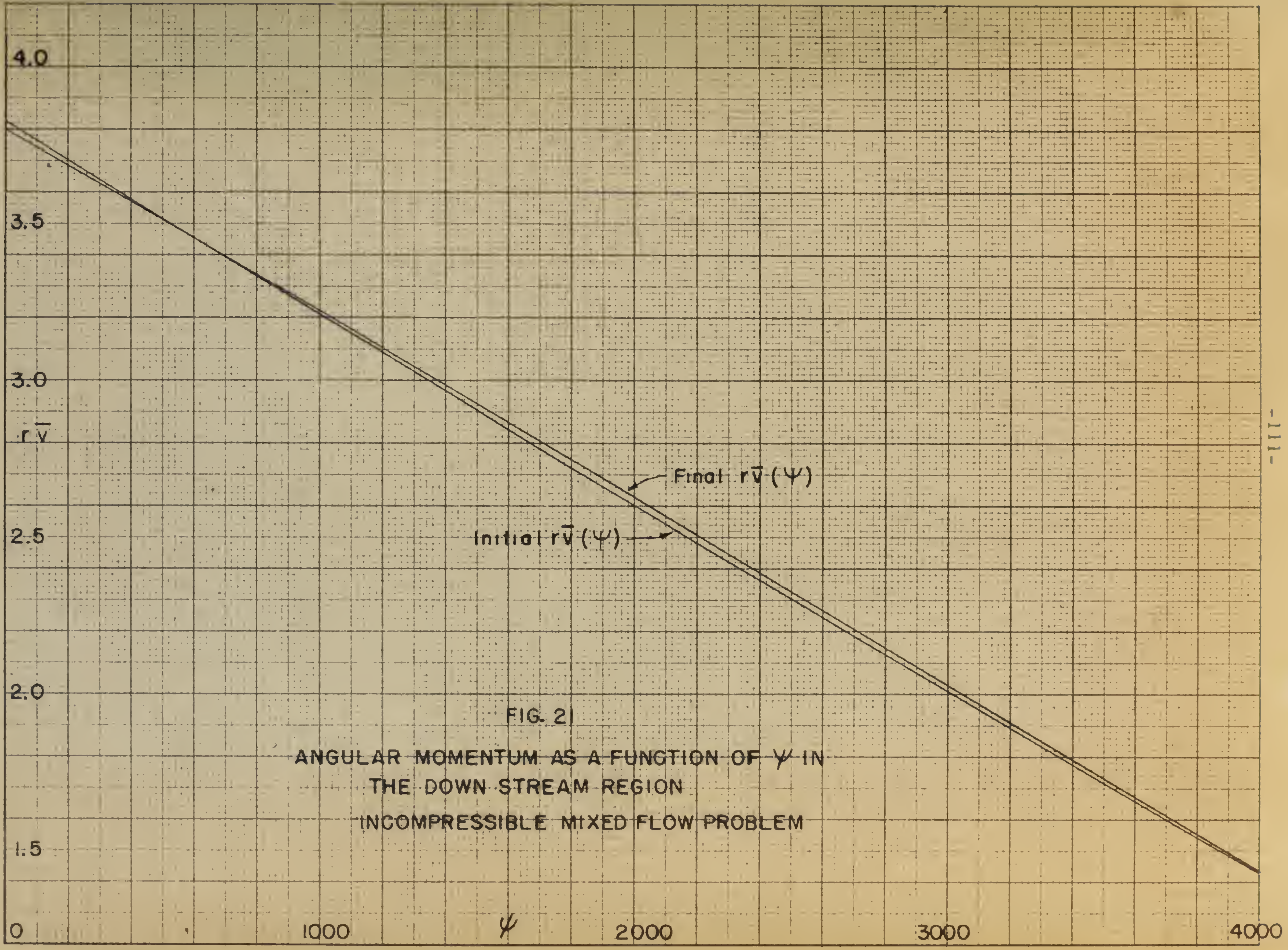


FIG. 21

ANGULAR MOMENTUM AS A FUNCTION OF ψ IN THE DOWN STREAM REGION

INCOMPRESSIBLE MIXED FLOW PROBLEM

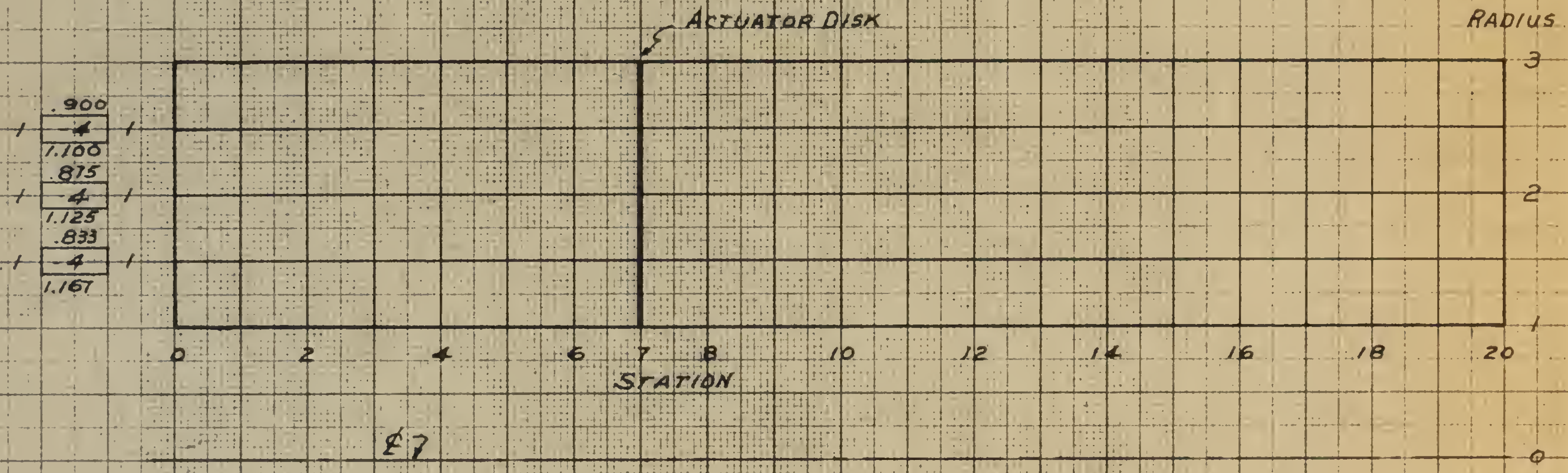


FIG. 22
INFLUENCE COEFFICIENTS
AT EACH POINT OF THE NET

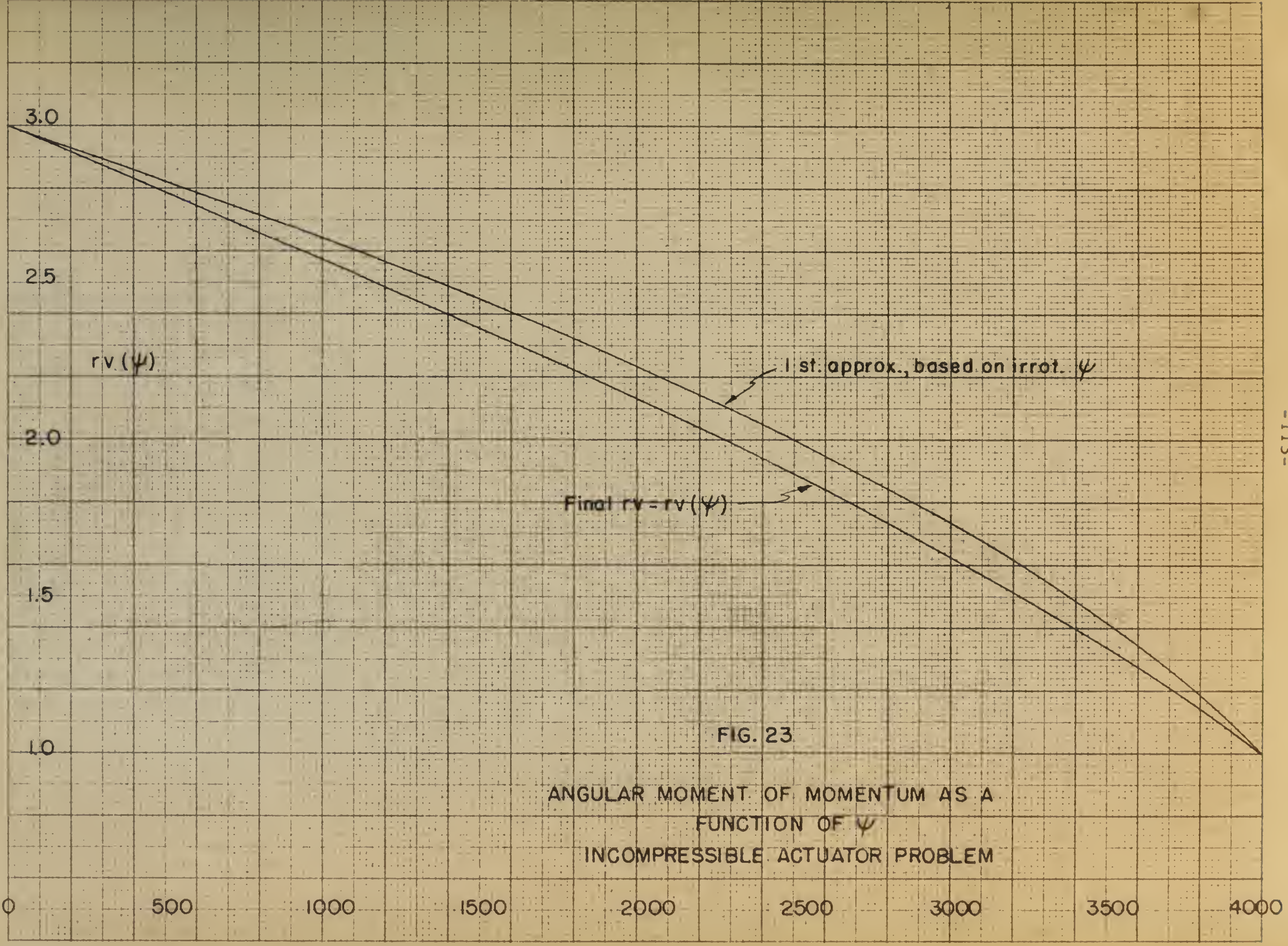


FIG. 23

ANGULAR MOMENT OF MOMENTUM AS A FUNCTION OF ψ
INCOMPRESSIBLE ACTUATOR PROBLEM

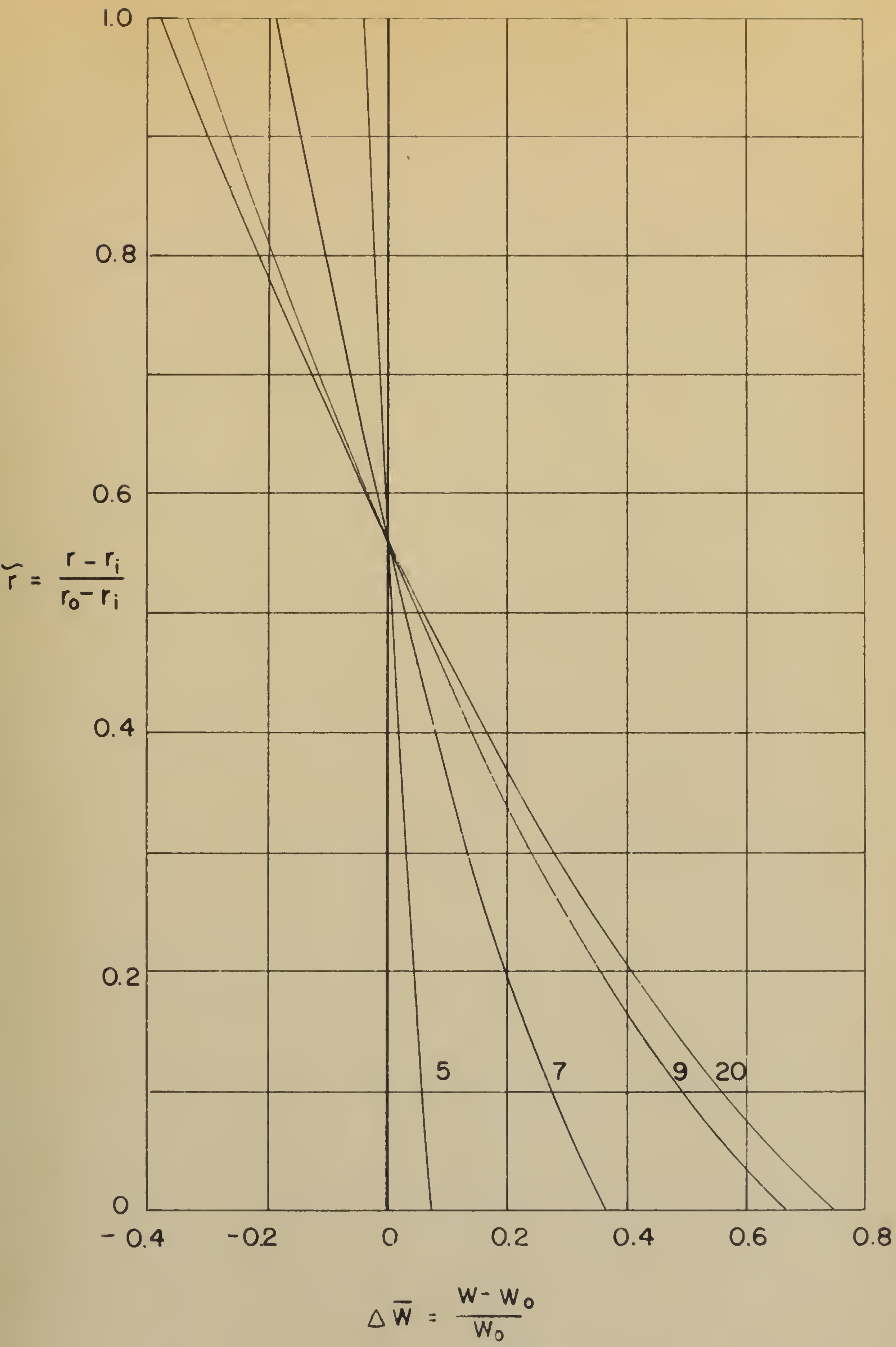


FIG. 24

AXIAL VELOCITY AT REPRESENTATIVE STATIONS

INCOMPRESSIBLE ACTUATOR PROBLEM

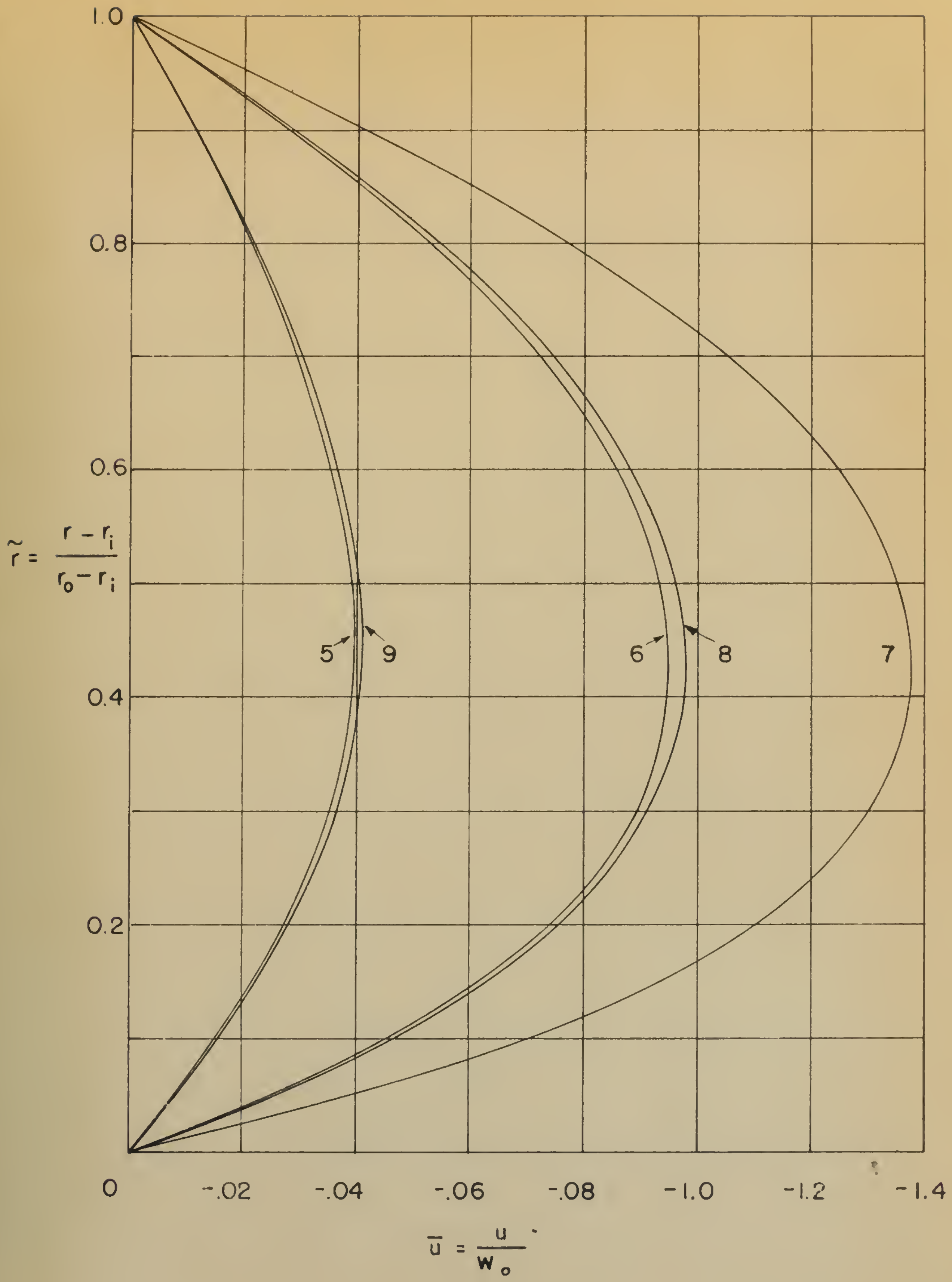


FIG. 25

RADIAL VELOCITY AT REPRESENTATIVE STATIONS

INCOMPRESSIBLE ACTUATOR PROBLEM

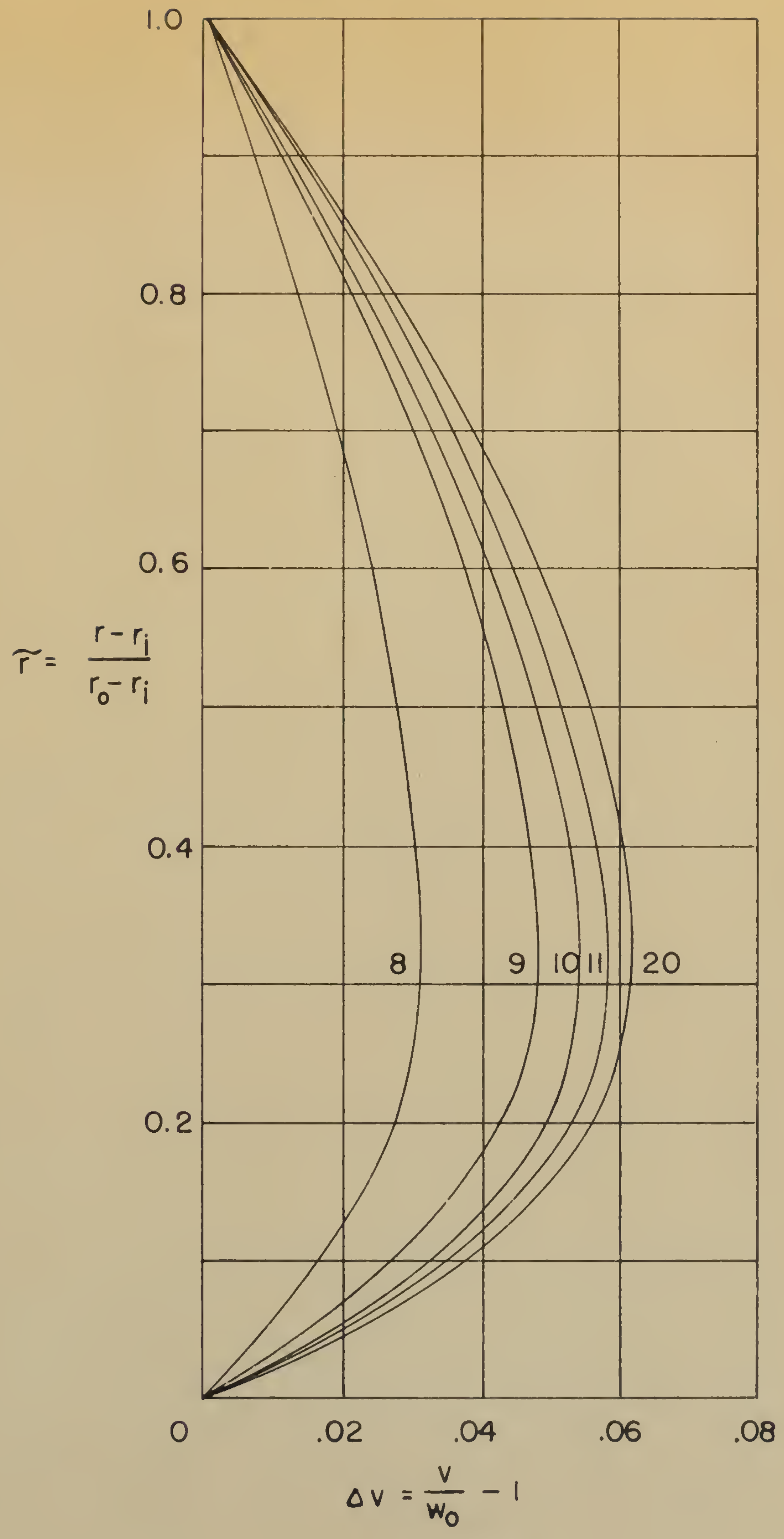


FIG. 26

TANGENTIAL VELOCITY AT REPRESENTATIVE STATIONS

INCOMPRESSIBLE ACTUATOR PROBLEM

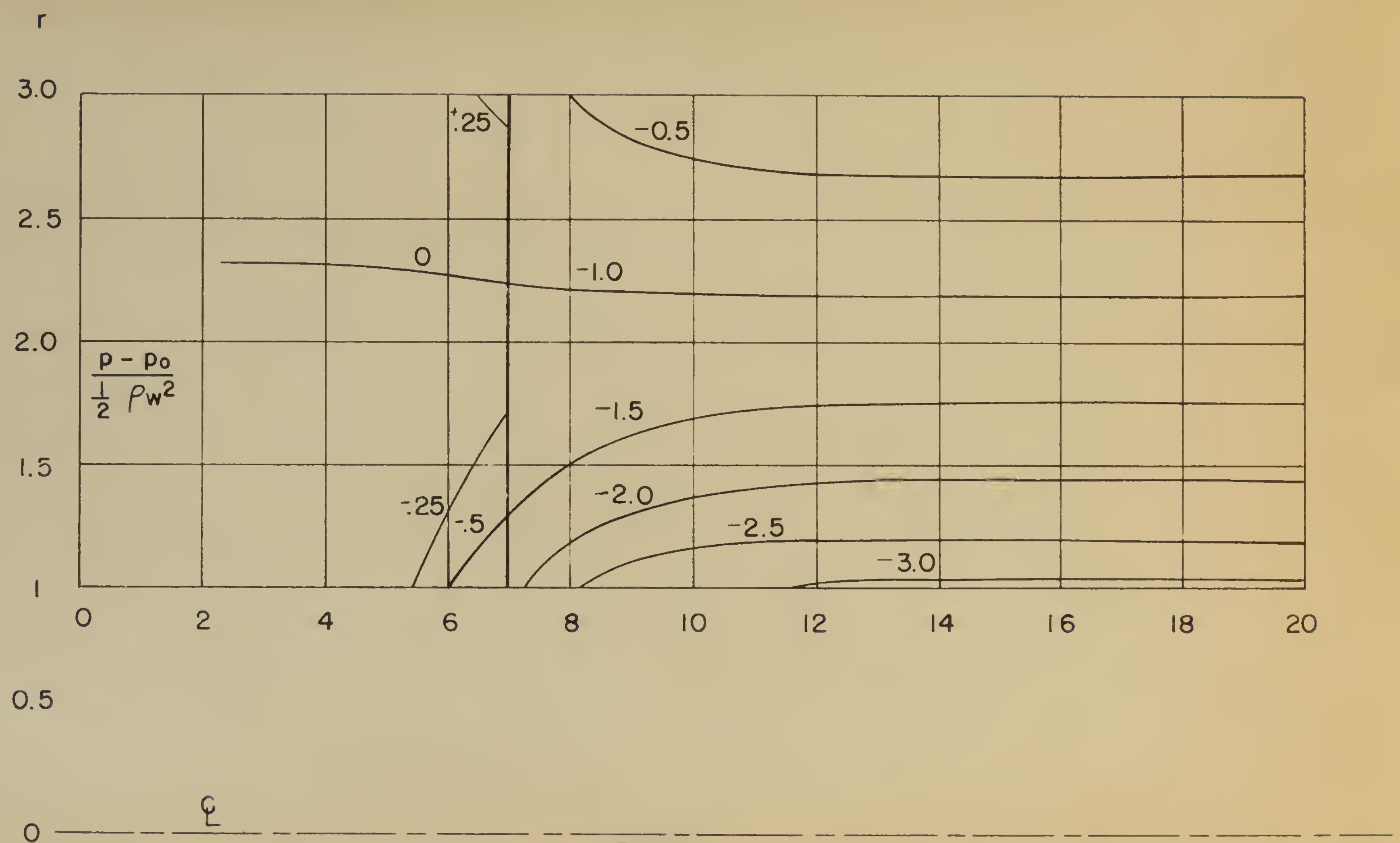


FIG. 27

LINES OF CONSTANT PRESSURE

INCOMPRESSIBLE ACTUATOR PROBLEM

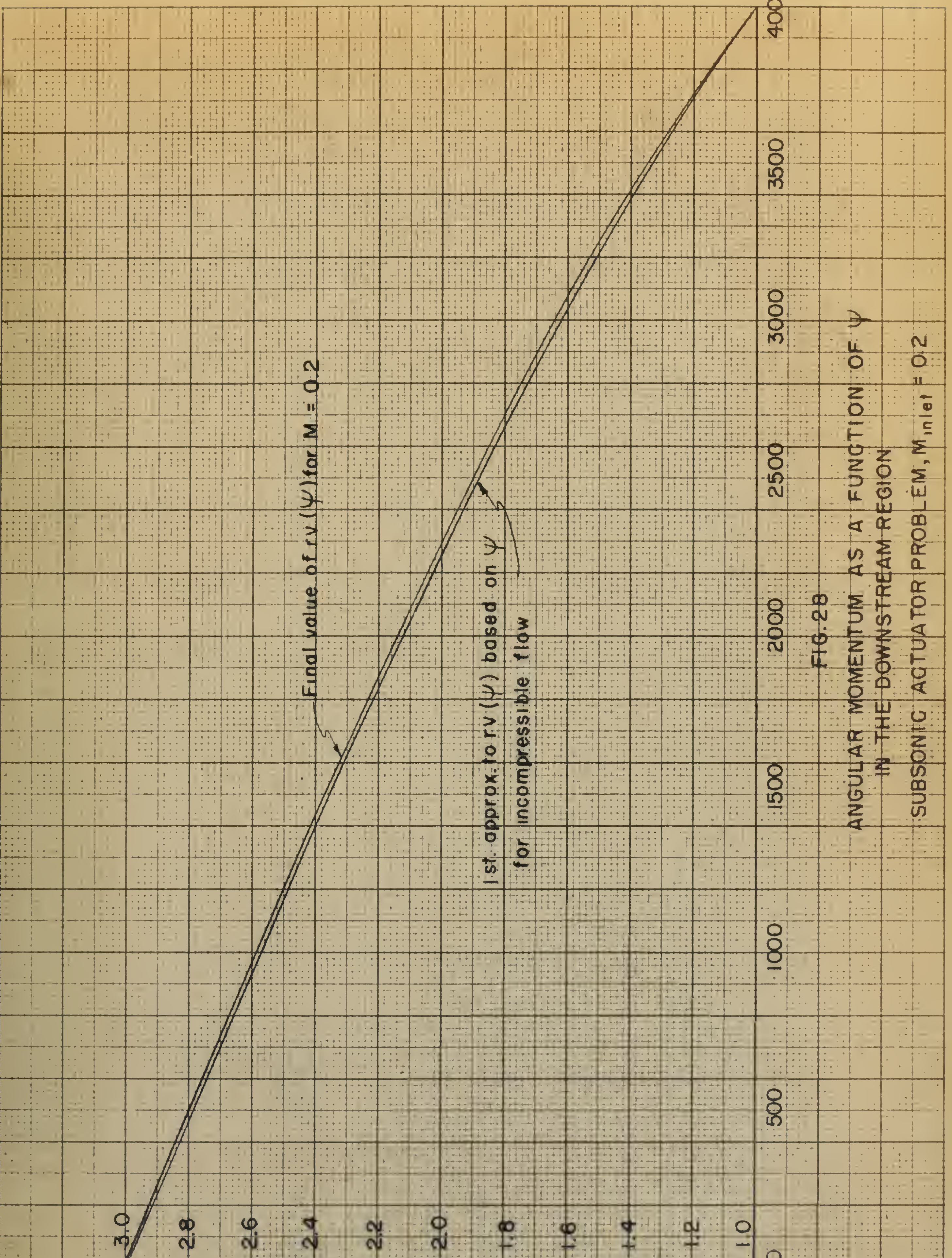
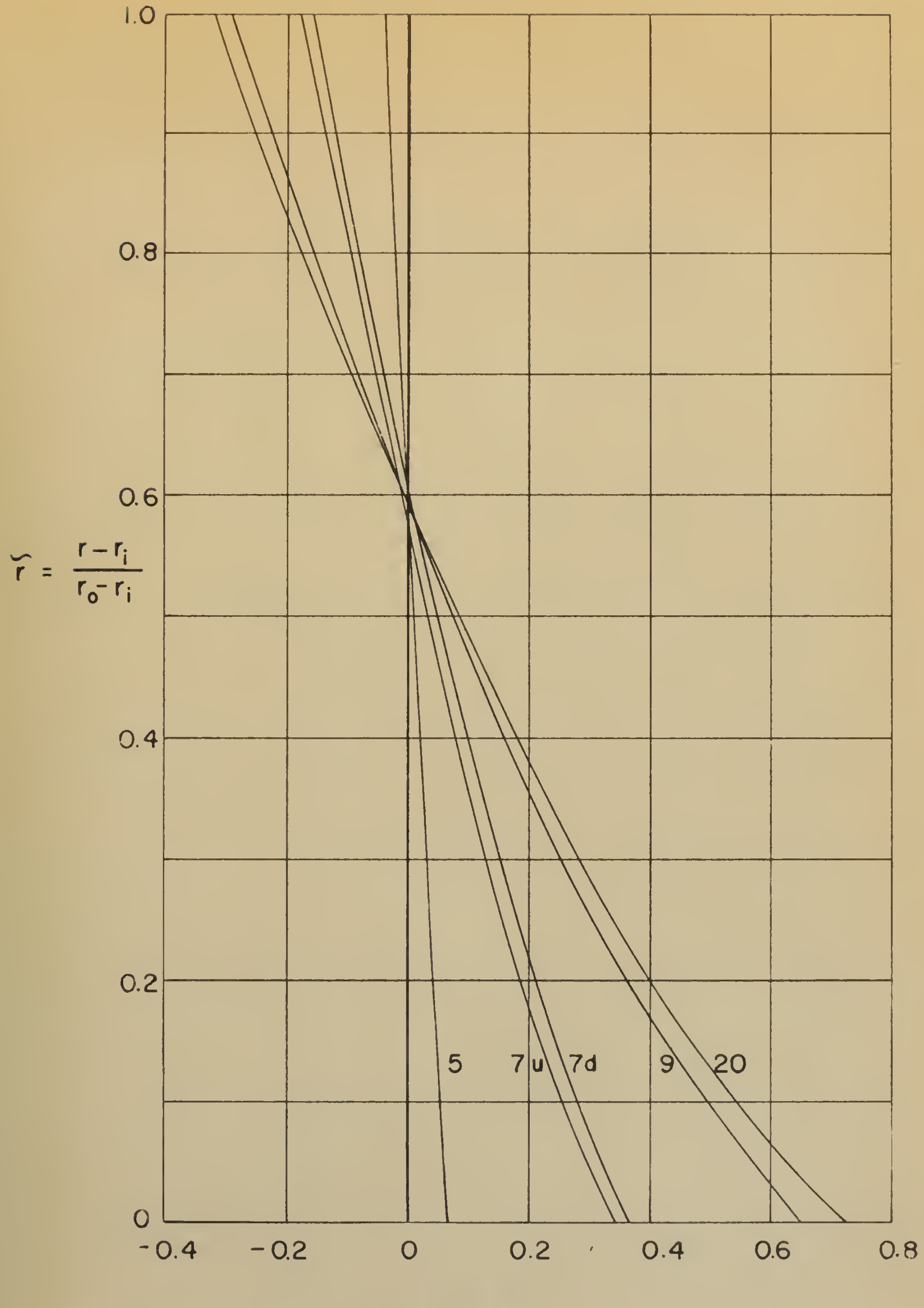


FIG. 28

ANGULAR MOMENTUM AS A FUNCTION OF ψ
IN THE DOWNSTREAM REGION

SUBSONIC ACTUATOR PROBLEM, $M_{inlet} = 0.2$



$\Delta \overline{w} = \frac{w - w_0}{w_0}$

FIG. 29

AXIAL VELOCITY

SUBSONIC ACTUATOR PROBLEM

$M_{inlet} = 0.2$

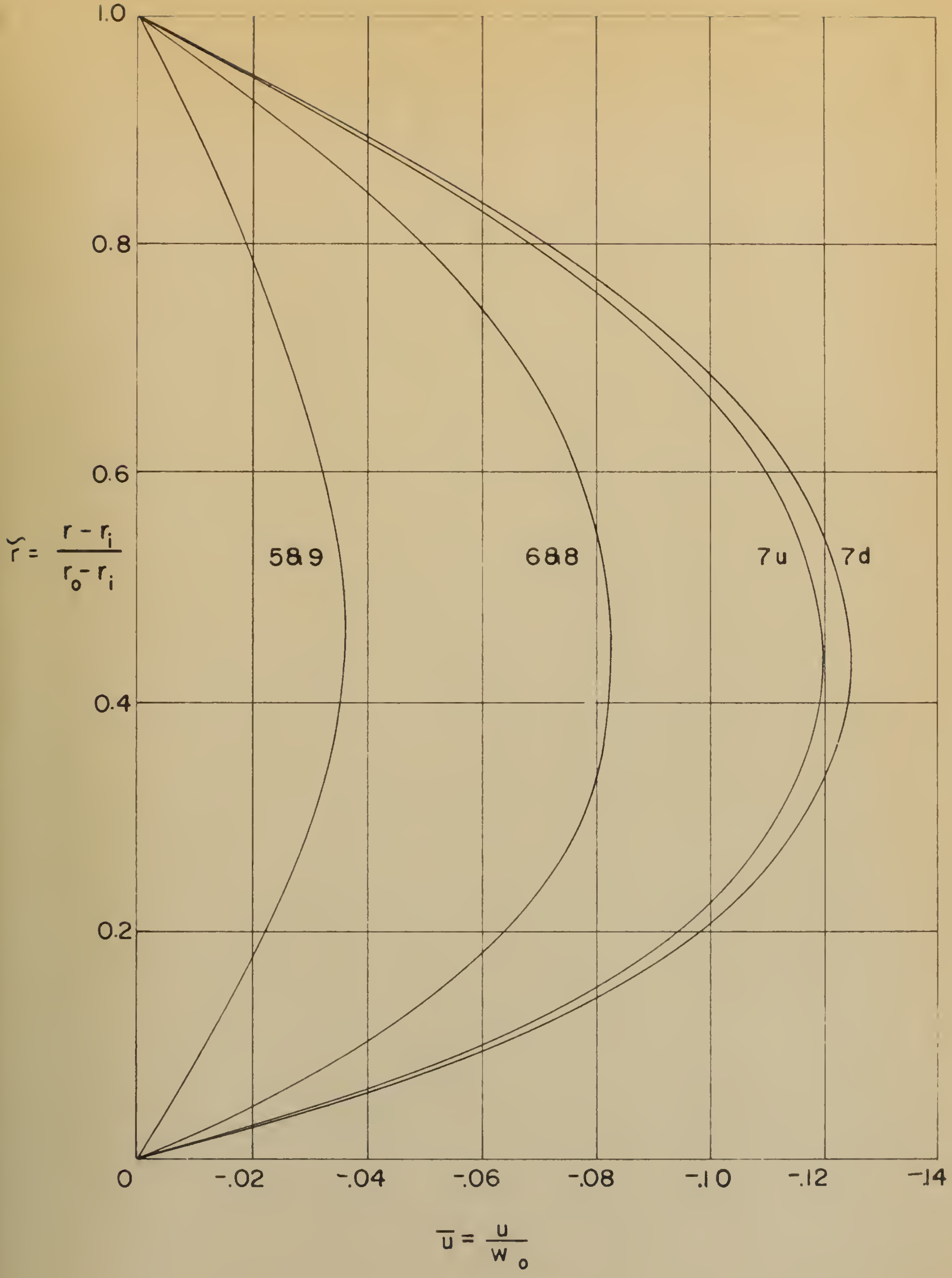


FIG. 30

RADIAL VELOCITY
SUBSONIC ACTUATOR PROBLEM

$M_{inlet} = 0.2$

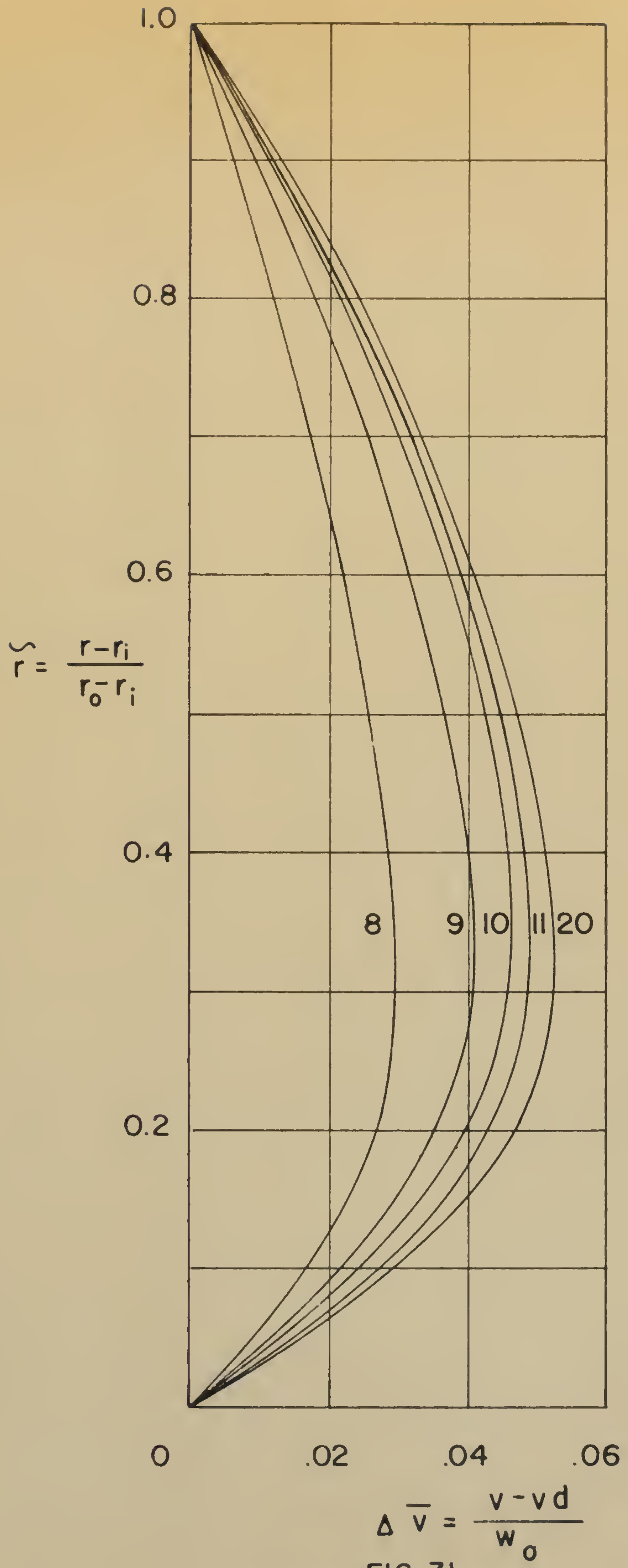


FIG. 31

TANGENTIAL VELOCITY

SUBSONIC ACTUATOR PROBLEM

$M_{inlet} = 0.2$

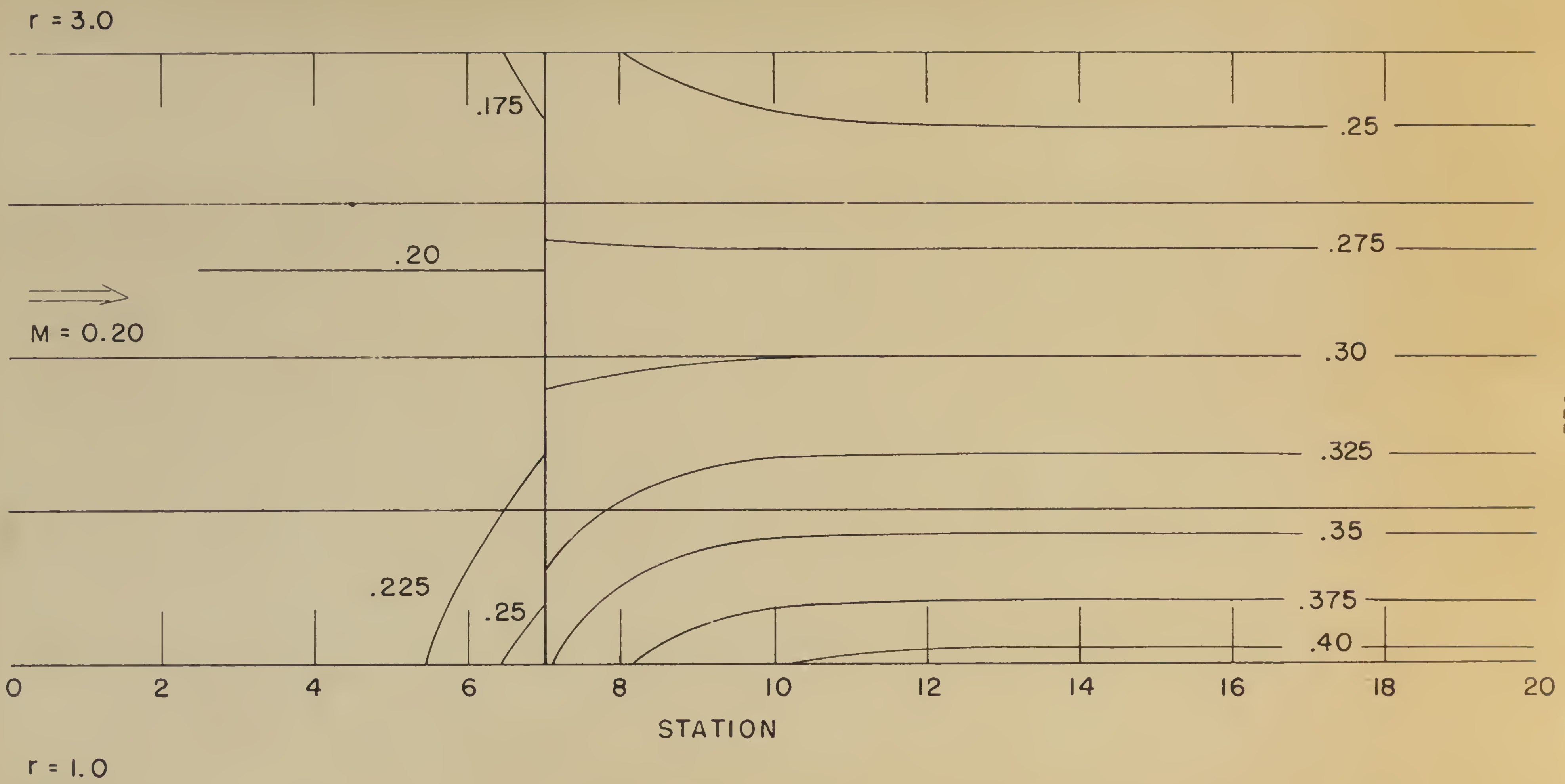


FIG. 32

MACH NO. BASED ON TOTAL VELOCITY
SUBSONIC ACTUATOR PROBLEM
$M_{inlet} = 0.2$

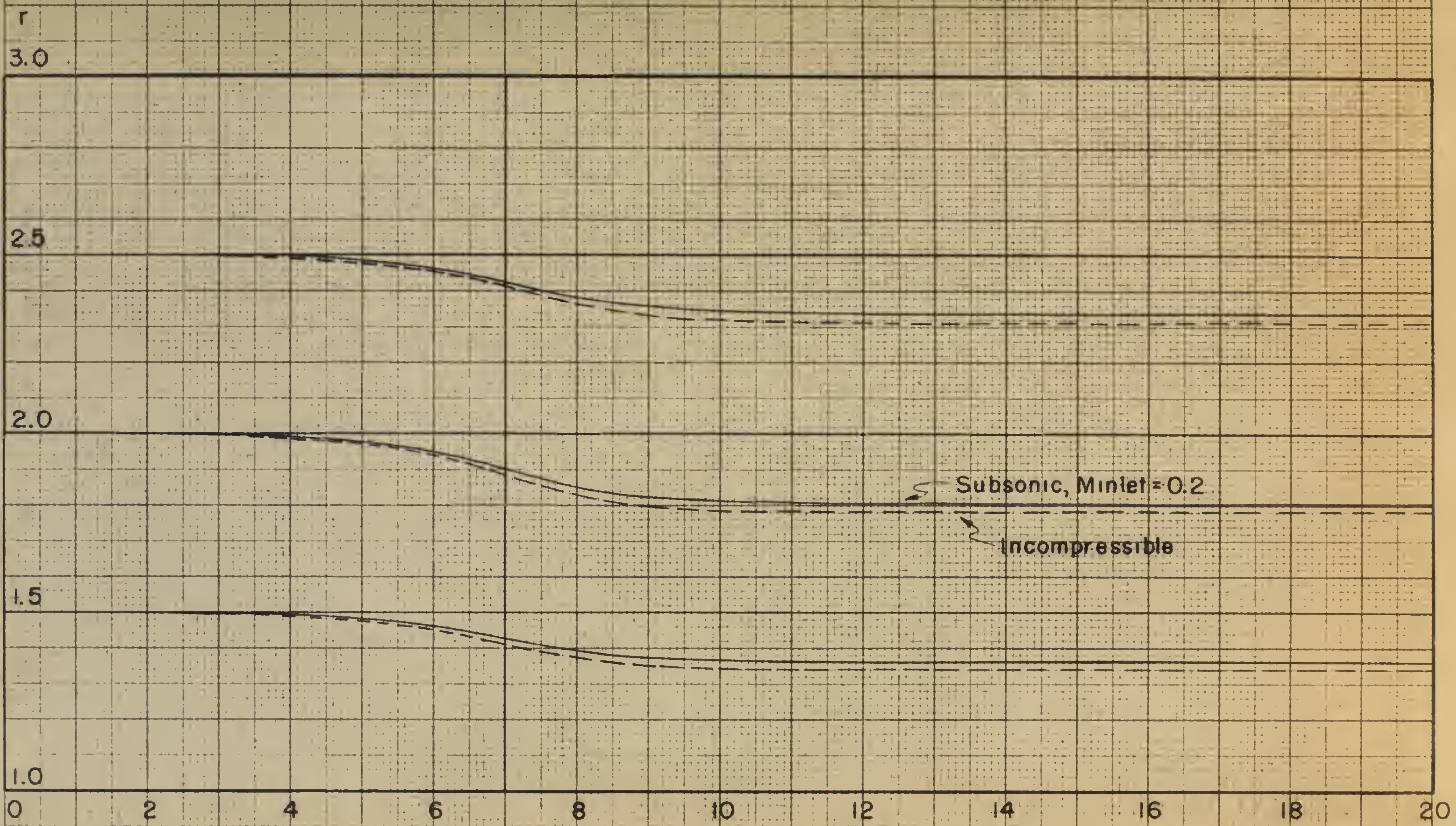


FIG. 33

COMPARISON OF STREAMLINES FOR INCOMPRESSIBLE
AND SUBSONIC ACTUATOR PROBLEMS

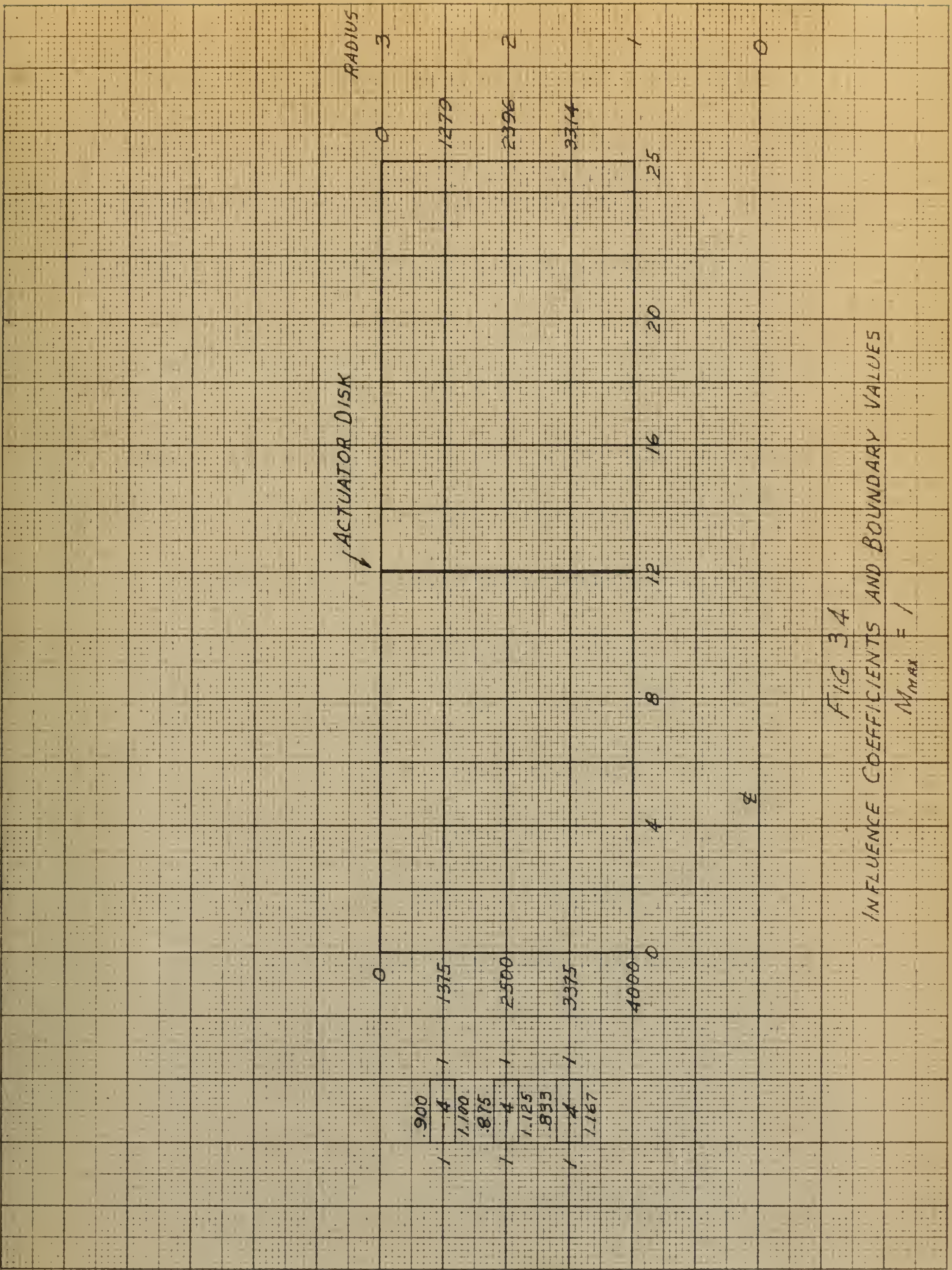


FIG 34
INFLUENCE COEFFICIENTS AND BOUNDARY VALUES
$M_{max} = 1$

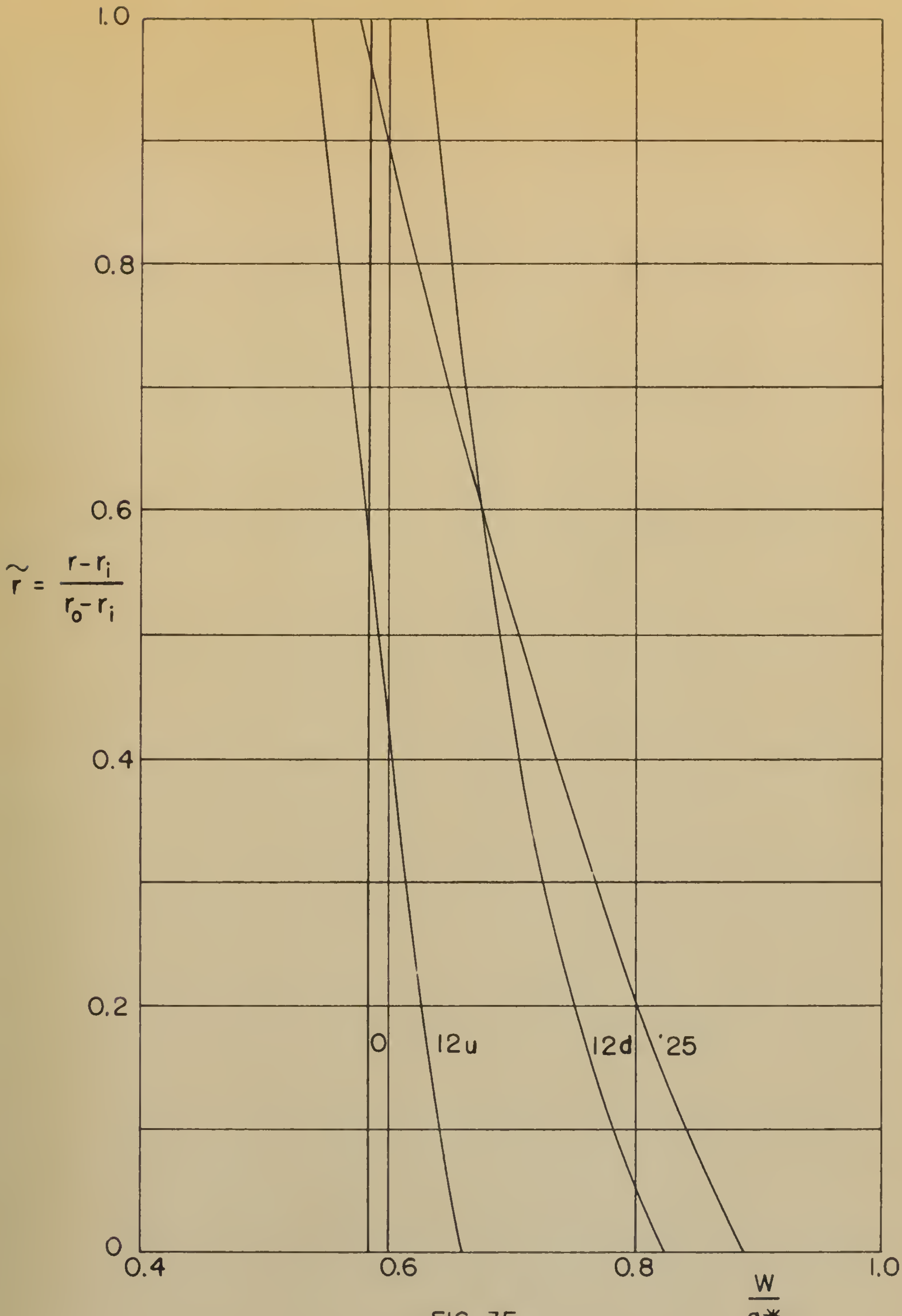


FIG. 35

AXIAL VELOCITY

TRANSONIC ACTUATOR PROBLEM $M_{max} = 1$

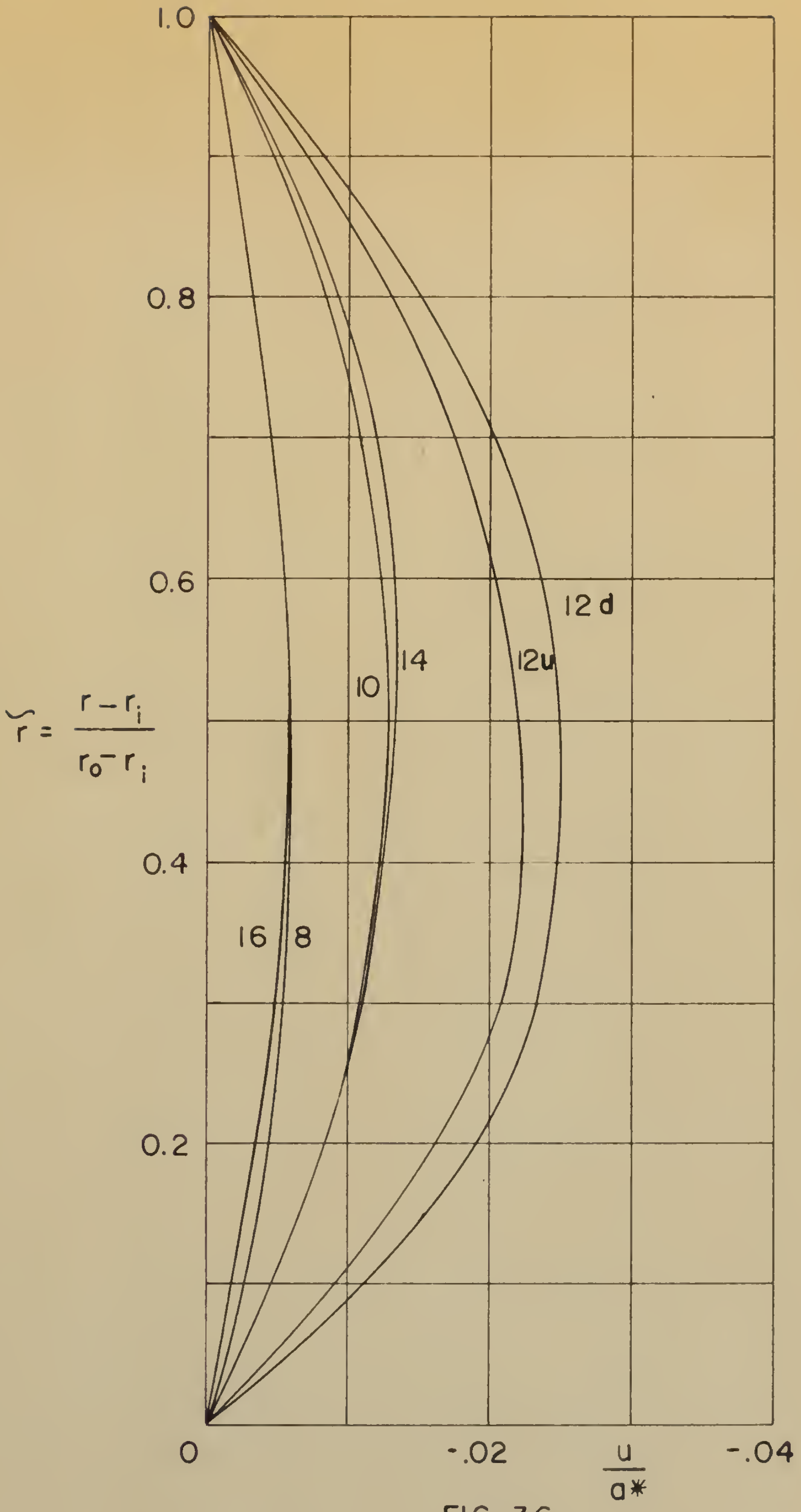


FIG. 36

RADIAL VELOCITY

TRANSONIC ACTUATOR PROBLEM

$M_{max} = 1$

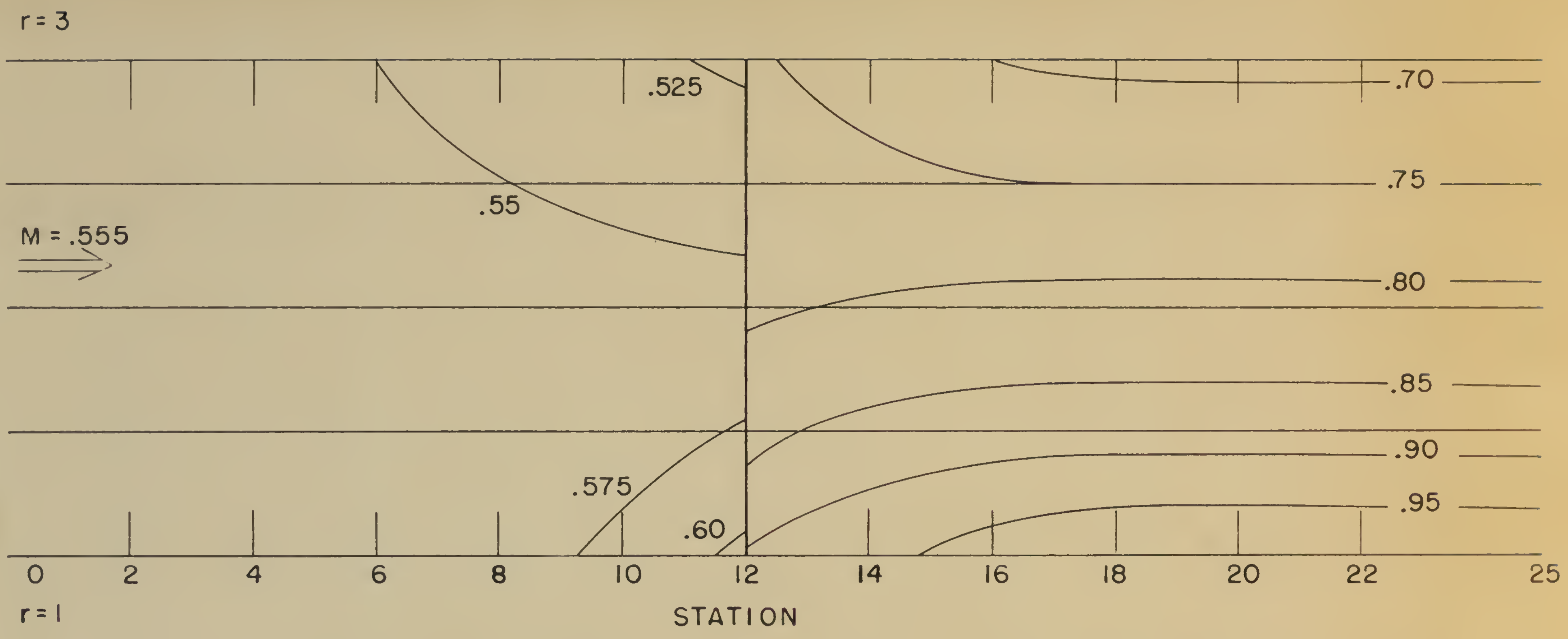


FIG. 37

MACH NO. BASED ON TOTAL VELOCITY

TRANSONIC ACTUATOR PROBLEM

$M_{max.} = 1$

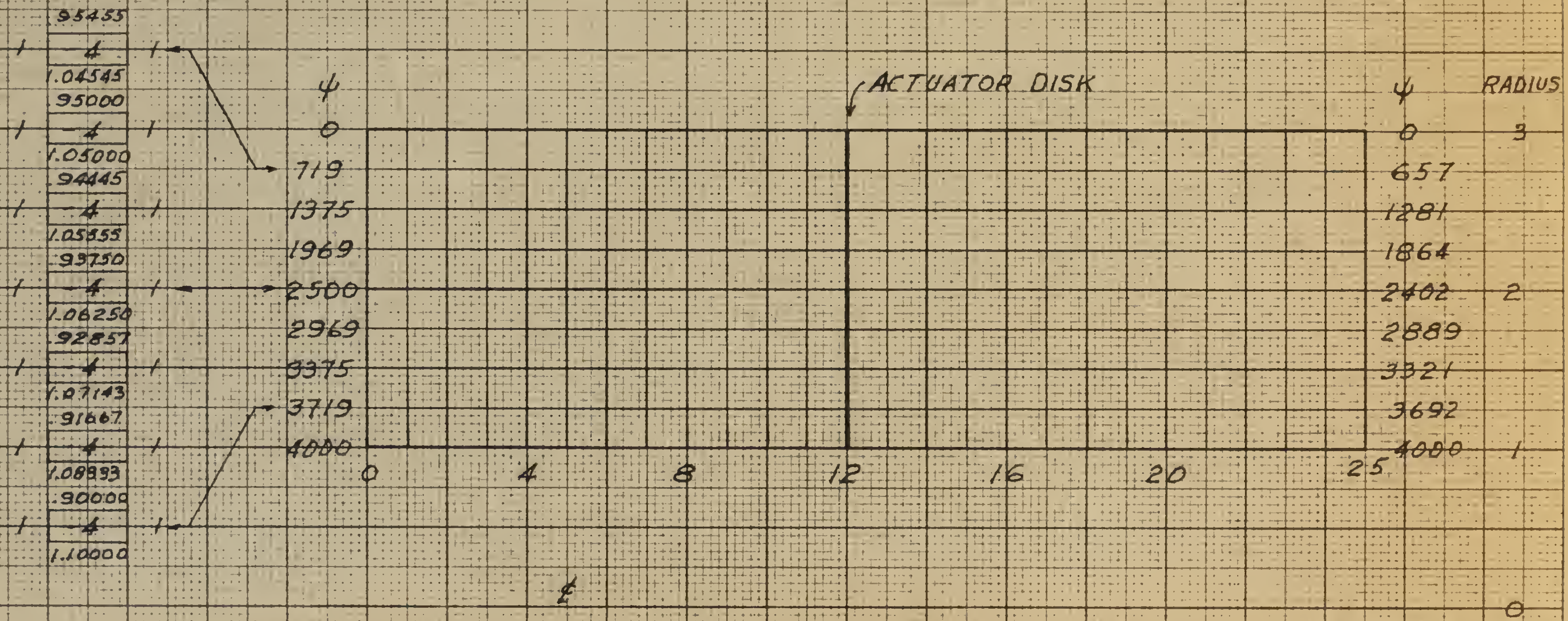


FIG. 38
INFLUENCE COEFFICIENTS AND BOUNDARY VALUES
$M_{MAX} = 1.12$

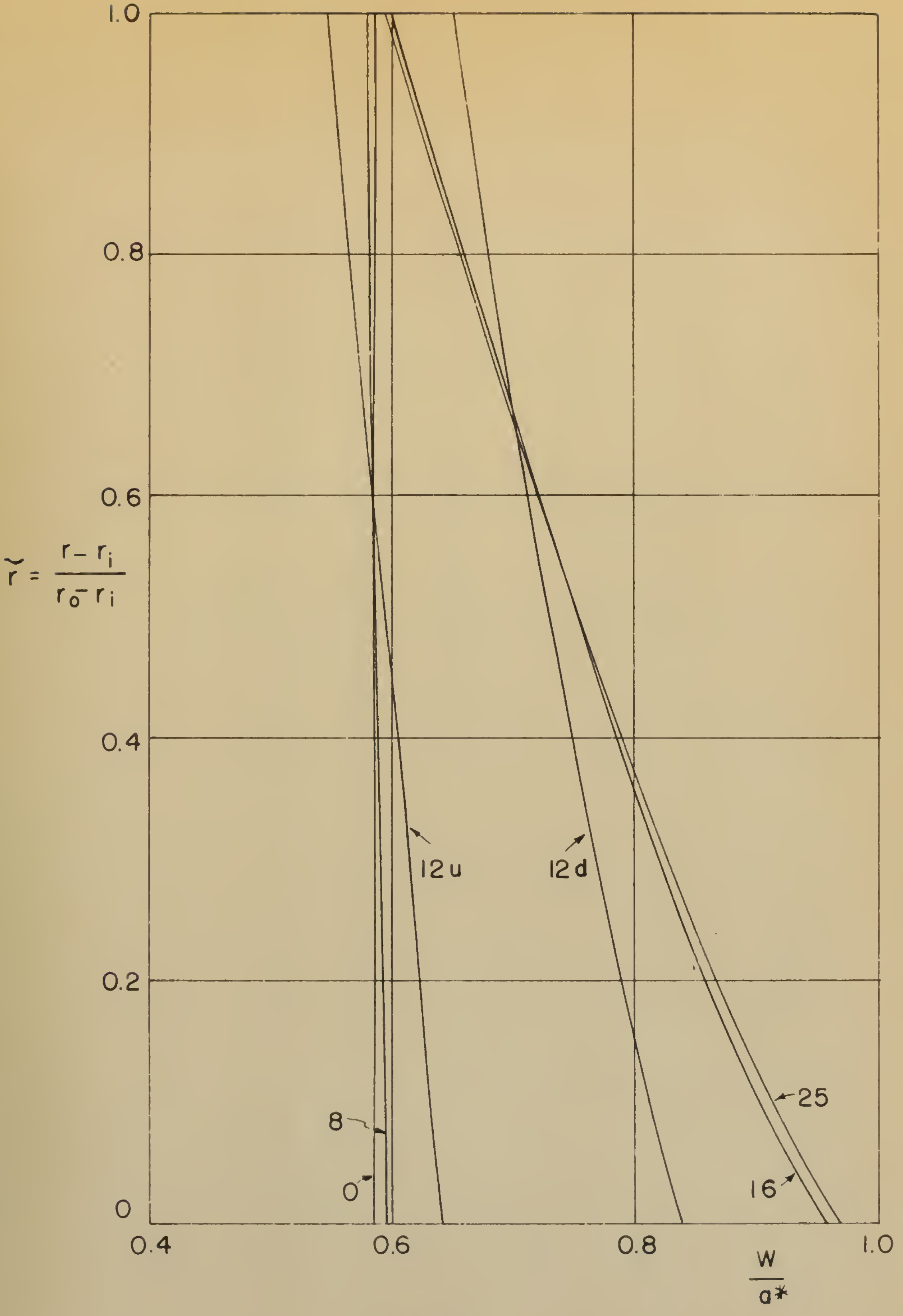


FIG. 39

AXIAL VELOCITY

$M_{max} = 1.12$

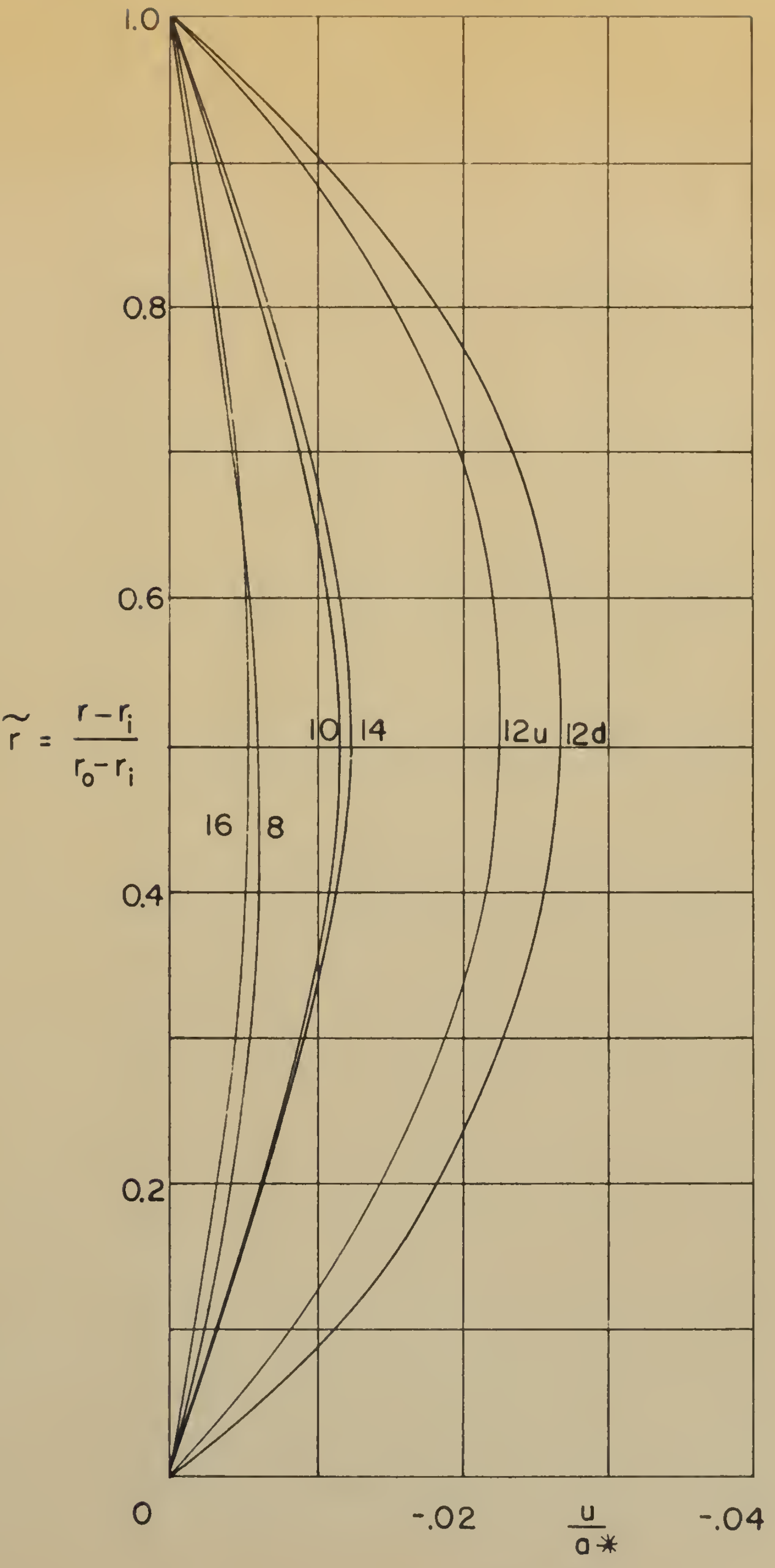


FIG. 40

RADIAL VELOCITY

TRANSONIC ACTUATOR PROBLEM

$M_{max} = 1.12$

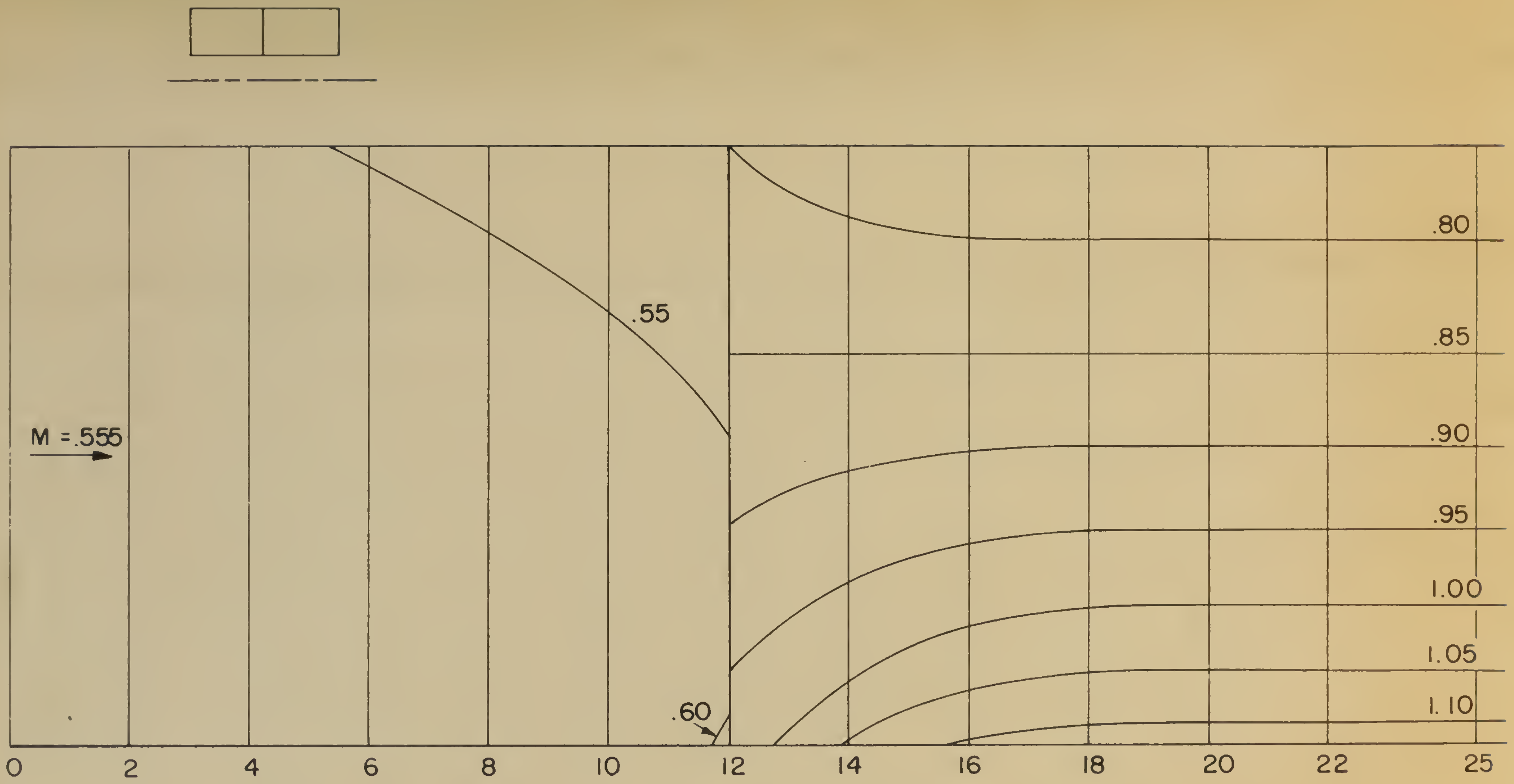


FIG. 41

MACH NUMBER OF TOTAL VELOCITY

$M_{max} = 1.12$

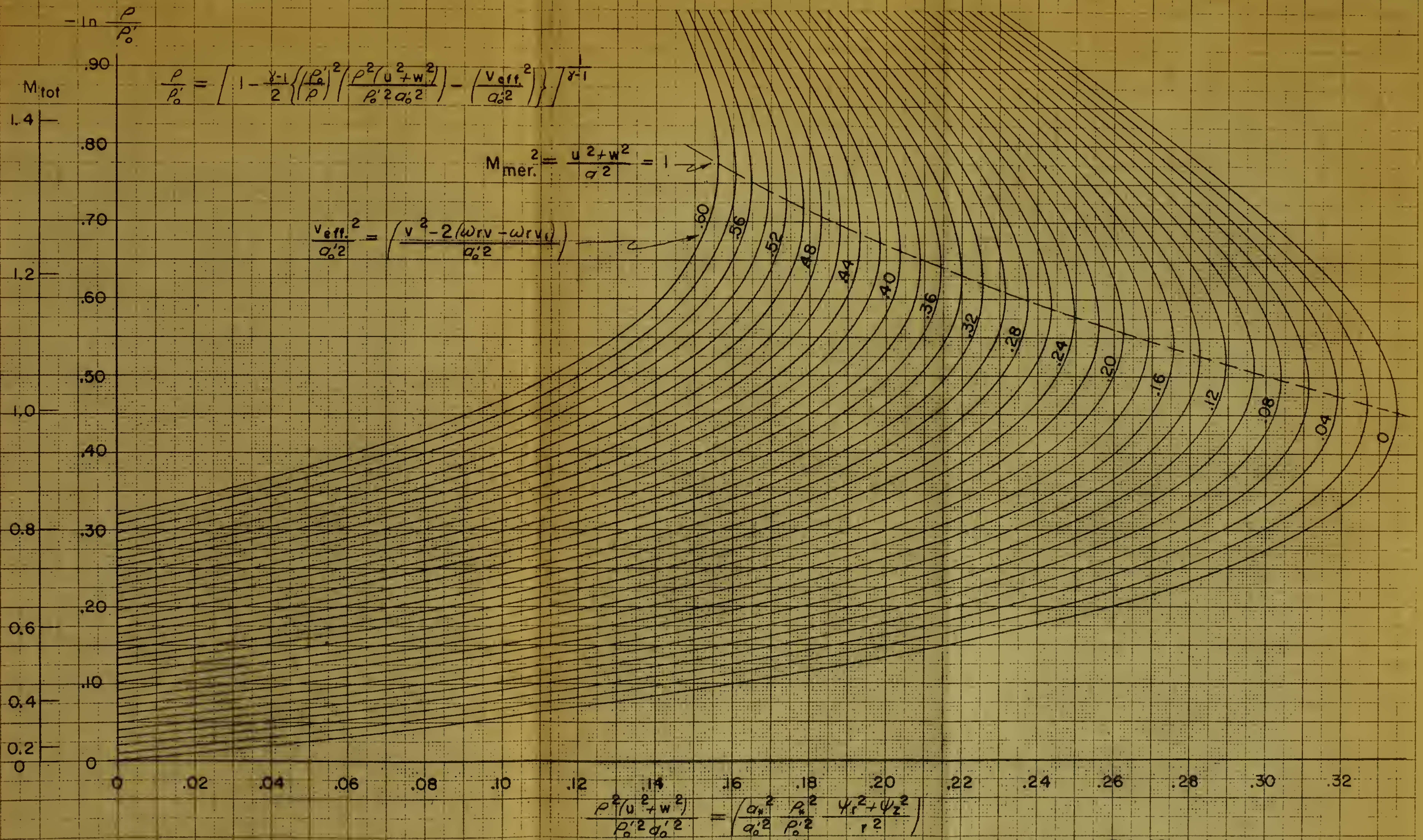


FIG. 42

DENSITY RATIO vs MASS FLOW AND
EFFECTIVE TANGENTIAL VELOCITY
~ FROM ISENTROPIC ENERGY RELATION

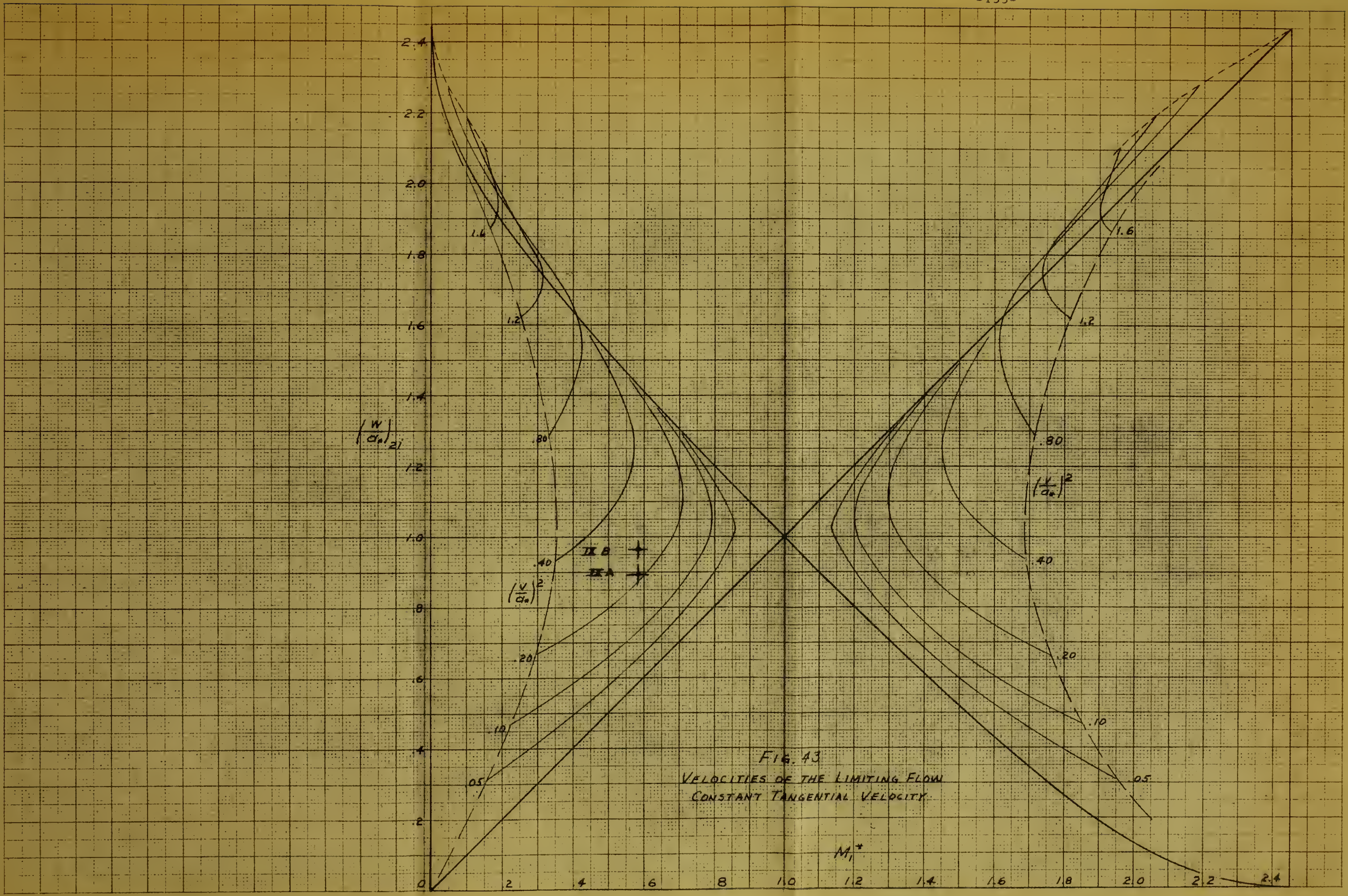


FIG. 43

VELOCITIES OF THE LIMITING FLOW

CONSTANT TANGENTIAL VELOCITY

DIETRICH POST CO. NO. 149 MILLIMETERS 380 BY 250 DIVISIONS
PRINTED IN U.S.A. ON CLEARPRINT TECHNICAL PAPER NO. 1000H

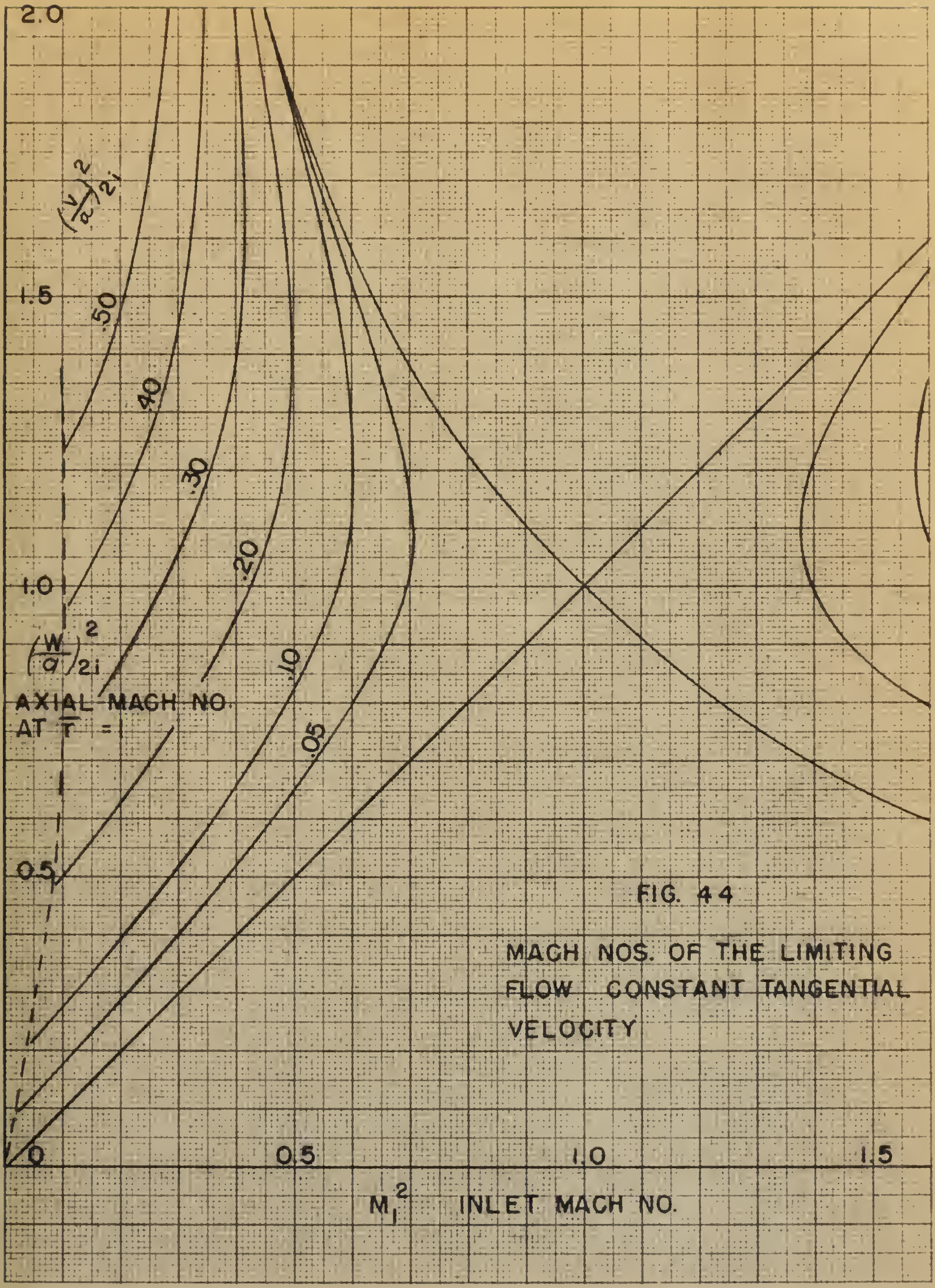


FIG. 44

MACH NOS. OF THE LIMITING FLOW CONSTANT TANGENTIAL VELOCITY

TABLE I

The Channel Boundaries and the Blade Shape Parameters for the Mixed Flow Example

Station	Hub Radius	Shroud Radius	Blade Shape Param.*, f_z
0-6	1.0000	3.0000	--
7	1.0000	3.0000	-.371
7.5	1.0000	3.0000	-.364
8	1.0000	3.0000	-.350
8.5	1.0038	3.0009	-.328
9	1.0288	3.0072	-.300
9.5	1.0908	3.0227	-.270
10	1.1955	3.0489	-.238
10.5	1.3371	3.0843	-.202
11	1.5000	3.1250	-.166
11.5	1.6629	3.1659	-.132
12	1.8045	3.2011	-.098
12.5	1.9091	3.2273	-.072
13	1.9712	3.2428	-.052
13.5	1.9962	3.2491	-.038
14	2.0000	3.2500	-.032
15-25	2.0000	3.2500	--

---

* $f_r = 0$

TABLE II

Limiting Values Far Downstream for Transonic Actuator Example with $M_{max} = 1.12$

r	$\frac{a^2}{a_*^2}$	$\frac{\rho}{\rho_*}$	$\frac{1}{\rho}\frac{\partial\rho}{\partial r}$	$\frac{w}{a_*}$	$-\bar{\psi}_r$	$\frac{R}{\bar{\sigma}^2}$	$\bar{\psi}$
1	.9607	.9046	.27480	.9658	.8737	.0072	3.1872
1.125	.9731	.9341	.24176	.9330	.9805	.0278	
1.25	.9843	.9612	.21456	.9027	1.0846	.0484	2.9422
1.375	.9943	.9858	.19310	.8744	1.1852	.0688	
1.5	1.0035	1.0088	.17539	.8476	1.2826	.0892	2.6460
1.625	1.0120	1.0303	.16054	.8224	1.3769	.1098	
1.75	1.0198	1.0503	.14793	.7982	1.4671	.1304	2.3200
1.875	1.0271	1.0691	.13709	.7751	1.5538	.1512	
2	1.0339	1.0869	.12767	.7527	1.6362	.1724	1.9137
2.125	1.0403	1.1038	.11942	.7311	1.7149	.1937	
2.25	1.0463	1.1198	.11214	.7102	1.7893	.2156	1.4852
2.375	1.0520	1.1351	.10566	.6899	1.8599	.2379	
2.5	1.0575	1.1500	.09985	.6700	1.9262	.2608	1.0204
2.625	1.0626	1.1639	.09464	.6506	1.9877	.2841	
2.75	1.0675	1.1774	.08993	.6312	2.0437	.3086	.5236
2.875	1.0722	1.1904	.08564	.6123	2.0955	.3338	
3.0	1.0767	1.2029	.08173	.5937	2.1425	.3599	0

TABLE III

Tabulated Values of the Isentropic Energy Relation
Density Ratio vs. Mass Flow and Effective Tangential Velocity

$$\left(\frac{\rho}{\rho_0'}\right)^2 \frac{u^2+w^2}{a_0'^2}$$

$\frac{v_{eff.}^2}{a_0'^2}$ $-\ln\frac{\rho}{\rho_0'}$	0	.0012	.0046	.0110	.0205	.0330	.0500	.0715	.0995
0	0	.0025	.0100	.0225	.0400	.0625	.0900	.1225	.1600
.02				.0029	.0208	.0438	.0719	.1052	.1436
.04					.0016	.0251	.0538	.0878	.1272
.06						.0063	.0357	.0705	.1108
.08							.0176	.0532	.0944
.10								.0358	.0781
.12								.0185	.0617
.14								.0011	.0453
.16									.0289
.18									.0125
.20									
.22									
.24									
.26									
.28									
.30									
.32									
.34									
.36									
.38									
.40									
.42									
.44									
.46									
.48									
.50									
.52									
.54									
.56									
.58									
.60									

TABLE III (Cont'd)

$$\left(\frac{\rho}{\rho_0'}\right)^2 \frac{u^2+w^2}{a_0'^2}$$

$\frac{V_{eff.}^2}{a_0'^2}$ \ $-\ln\frac{\rho}{\rho_0'}$	.1132	.1290	.1458	.1655	.1882	.2022	.2173	.2350	.2550
0	.1764	.1936	.2116	.2304	.2500	.2601	.2704	.2809	.2916
.02	.1605	.1782	.1967	.2160	.2363	.2468	.2574	.2684	.2796
.04	.1445	.1627	.1817	.2017	.2226	.2334	.2444	.2559	.2676
.06	.1286	.1473	.1668	.1845	.2089	.2201	.2314	.2434	.2556
.08	.1126	.1318	.1518	.1729	.1952	.2067	.2185	.2309	.2436
.10	.0967	.1164	.1369	.1586	.1814	.1934	.2055	.2184	.2315
.12	.0807	.1009	.1220	.1437	.1677	.1800	.1925	.2059	.2195
.14	.0648	.0855	.1070	.1298	.1540	.1667	.1795	.1934	.2075
.16	.0488	.0700	.0921	.1155	.1403	.1533	.1665	.1809	.1955
.18	.0329	.0546	.0771	.1011	.1266	.1400	.1535	.1685	.1835
.20	.0169	.0391	.0622	.0867	.1129	.1266	.1405	.1560	.1715
.22	.0010	.0237	.0473	.0724	.0992	.1133	.1276	.1435	.1595
.24		.0082	.0323	.0580	.0855	.0999	.1146	.1310	.1475
.26			.0174	.0436	.0717	.0866	.1036	.1185	.1354
.28			.0024	.0293	.0580	.0733	.0886	.1060	.1234
.30				.0149	.0443	.0599	.0756	.0935	.1114
.32				.0005	.0306	.0406	.0626	.0810	.0994
.34					.0169	.0332	.0466	.0685	.0874
.36					.0032	.0199	.0367	.0560	.0754
.38						.0065	.0237	.0435	.0634
.40							.0107	.0310	.0514
.42								.0185	.0393
.44								.0015	.0273
.46									.0153
.48									.0033
.50									
.52									
.54									
.56									
.58									
.60									

TABLE III (Cont'd)

$$\left(\frac{\rho}{\rho_0'}\right)^2 \frac{u^2+w^2}{a_0'^2}$$

$\frac{V_{eff.}^2}{a_0'^2}$ $-\ln\frac{\rho}{\rho_0'}$	.2800	.3090	.3300	.3525	.3836	.4550	.5350	.5800	.6140
0	.3025	.3136	.3192	.3249	.3306	.3345	.3306	.3249	.3192
.02	.2911	.3028	.3089	.3150	.3213	.3264	.3237	.3186	.3134
.04	.2796	.2920	.2985	.3051	.3120	.3184	.3169	.3124	.3075
.06	.2682	.2813	.2882	.2952	.3027	.3103	.3100	.3061	.3017
.08	.2568	.2705	.2778	.2854	.2934	.3023	.3032	.2998	.2958
.10	.2453	.2597	.2675	.2755	.2841	.2942	.2963	.2936	.2900
.12	.2339	.2489	.2572	.2656	.2748	.2862	.2895	.2873	.2841
.14	.2225	.2382	.2468	.2557	.2655	.2781	.2826	.2811	.2783
.16	.2111	.2274	.2365	.2458	.2562	.2701	.2758	.2748	.2725
.18	.1996	.2166	.2261	.2359	.2969	.2620	.2689	.2685	.2666
.20	.1882	.2058	.2158	.2261	.2376	.2540	.2621	.2623	.2608
.22	.1768	.1951	.2055	.2162	.2283	.2459	.2552	.2560	.2549
.24	.1653	.1843	.1951	.2063	.2190	.2379	.2484	.2497	.2491
.26	.1539	.1735	.1848	.1964	.2097	.2298	.2415	.2435	.2433
.28	.1425	.1627	.1744	.1865	.2004	.2218	.2347	.2372	.2374
.30	.1310	.1520	.1641	.1766	.1911	.2137	.2278	.2309	.2316
.32	.1196	.1412	.1538	.1668	.1818	.2057	.2210	.2247	.2257
.34	.1082	.1304	.1434	.1569	.1725	.1976	.2141	.2184	.2199
.36	.0968	.1196	.1331	.1470	.1632	.1896	.2073	.2121	.2140
.38	.0853	.1089	.1227	.1371	.1539	.1815	.2004	.2059	.2082
.40	.0739	.0981	.1124	.1272	.1446	.1735	.1936	.1996	.2024
.42	.0625	.0873	.1021	.1173	.1353	.1654	.1867	.1934	.1965
.44	.0510	.0765	.0917	.1075	.1260	.1574	.1799	.1871	.1907
.46	.0396	.0658	.0814	.0976	.1167	.1493	.1730	.1808	.1848
.48	.0282	.0550	.0710	.0877	.1074	.1413	.1662	.1746	.1790
.50	.0167	.0440	.0607	.0778	.0981	.1332	.1593	.1683	.1731
.52	.0053	.0334	.0504	.0679	.0889	.1251	.1525	.1620	.1673
.54		.0226	.0400	.0580	.0796	.1171	.1456	.1558	.1615
.56		.0119	.0297	.0481	.0703	.1090	.1388	.1495	.1556
.58		.0011	.0193	.0383	.0610	.1010	.1319	.1432	.1498
.60			.0090	.0284	.0517	.0929	.1251	.1370	.1439

TABLE III (Cont'd)

$$\left(\frac{p}{p_0'}\right)^2 \frac{u^2+w^2}{a_0'^2}$$

$\frac{v_{eff}^2}{a_0'^2}$ / $-\ln\frac{p}{p_0'}$	.6450	.6955	.741	.785	.824	.863	.902	.974
0	.3136	.3025	.2916	.2809	.2704	.2601	.2500	.2304
.02	.3081	.2975	.2871	.2767	.2666	.2565	.2467	.2276
.04	.3026	.2925	.2825	.2726	.2627	.2530	.2434	.2247
.06	.2971	.2875	.2780	.2684	.2589	.2494	.2402	.2219
.08	.2916	.2826	.2735	.2643	.2550	.2459	.2369	.2190
.10	.2860	.2776	.2689	.2601	.2512	.2423	.2336	.2162
.12	.2805	.2726	.2644	.2560	.2474	.2388	.2301	.2133
.14	.2750	.2676	.2599	.2518	.2435	.2352	.2270	.2105
.16	.2695	.2626	.2553	.2477	.2397	.2317	.2238	.2077
.18	.2640	.2576	.2508	.2435	.2358	.2281	.2205	.2048
.20	.2585	.2527	.2463	.2393	.2320	.2246	.2172	.2020
.22	.2530	.2477	.2417	.2352	.2282	.2210	.2139	.1991
.24	.2475	.2427	.2372	.2310	.2243	.2175	.2106	.1963
.26	.2419	.2377	.2327	.2268	.2205	.2139	.2074	.1935
.28	.2364	.2327	.2282	.2227	.2166	.2103	.2041	.1906
.30	.2309	.2277	.2236	.2185	.2128	.2068	.2008	.1878
.32	.2254	.2228	.2191	.2144	.2090	.2032	.1975	.1849
.34	.2199	.2178	.2146	.2102	.2051	.1997	.1942	.1821
.36	.2144	.2128	.2100	.2061	.2013	.1961	.1910	.1792
.38	.2089	.2078	.2055	.2019	.1974	.1926	.1877	.1764
.40	.2034	.2028	.2010	.1977	.1936	.1890	.1844	.1736
.42	.1978	.1978	.1967	.1936	.1896	.1855	.1811	.1707
.44	.1923	.1929	.1919	.1894	.1859	.1819	.1778	.1679
.46	.1868	.1879	.1874	.1853	.1821	.1784	.1746	.1650
.48	.1813	.1829	.1828	.1811	.1782	.1748	.1713	.1622
.50	.1758	.1779	.1783	.1769	.1743	.1712	.1673	.1593
.52	.1703	.1729	.1738	.1728	.1706	.1677	.1643	.1565
.54	.1648	.1679	.1692	.1686	.1667	.1641	.1614	.1531
.56	.1593	.1629	.1647	.1645	.1629	.1606	.1582	.1508
.58	.1538	.1580	.1602	.1603	.1590	.1570	.1549	.1480
.60	.1482	.1530	.1556	.1562	.1552	.1535	.1516	.1451

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